STOCHASTIC DIFFERENTIAL EQUATION FOR BROX DIFFUSION

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Abstract. This paper studies the weak and strong solutions to the stochastic differential equation $dX(t) = -\frac{1}{2}\dot{W}(X(t))dt + d\mathcal{B}(t)$, where $(\mathcal{B}(t), t \geq 0)$ is a standard Brownian motion and $W(x)$ is a two sided Brownian motion, independent of $\mathcal{B}$. It is shown that the Itô-McKean representation associated with any Brownian motion (independent of W) is a weak solution to the above equation. It is also shown that there exists a unique strong solution to the equation. Itô calculus for the solution is developed. For dealing with the singularity of drift term $\int_0^T \dot{W}(X(t))dt$, the main idea is to use the concept of local time together with the polygonal approximation W_π. Some new results on the local time of Brownian motion needed in our proof are established.

Contents

1. Introduction

Ever since the work of Sinai [21] on the random walk in random medium there has been a great amount of work on random processes in a random environment. One of the continuous time and continuous space analogues of Sinai's random walk is the Brownian motion in a white noise medium, namely, the Brox diffusion, which can be described briefly as follows. Let $(\mathcal{B}(t), t \geq 0)$ be a one dimensional standard Brownian motion and let $(W(x), x \in \mathbb{R})$ be a two sided one dimensional Brownian motion, independent of $\mathcal{B}$. Its derivative $\dot{W}(x)$ with respect to x in the sense of Schwartz distribution is called the white noise (see [9]). The Brox diffusion is a

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diffusion process $X(t)$ determined formally by the following stochastic differential equation

$$X(t) = -\frac{1}{2}\int_0^t \dot{W}(X(s))ds + \mathcal{B}(t)\,. \tag{1.1}$$

Throughout the paper, we assume the initial condition $X(0) = 0$ for simplicity. Since $\dot{W}$ is a distribution (generalized function), the conventional theory of stochastic differential equations does not apply to the above equation (1.1).

In the case W is nice (for example, $\dot{W}(x)$ is deterministic and globally Lipschitz continuous), then the solution $X(t)$ to (1.1) exists uniquely and it is a Markov process with generator

$$A = \frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right)\,. \tag{1.2}$$

In [3], the process $X(t)$ defined (formally) by (1.1) is identified as a Feller diffusion with the above generator A. The Itô-McKean's construction of this Feller diffusion from a Brownian motion via scale-transformation and time change is particularly used there. Let us briefly recall this construction. Let B be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, independent of $(W(x), x \in \mathbb{R})$ (Note that, if it is not stated otherwise, we assume throughout the paper that $(W(x), x \in \mathbb{R})$ is a two sided Brownian motion). We define the spatial transformation

$$S_W(x) = \int_0^x e^{W(z)}dz, \tag{1.3}$$

and the time change

$$T_{W,B}(t) = \int_0^t e^{-2W\circ S_W^{-1}(B(s))}ds\,. \tag{1.4}$$

Then, the Feller diffusion $(X(t)\,, t \ge 0)$ associated with (1.1) is represented as

$$X(t) = S_W^{-1}\circ B\circ T_{W,B}^{-1}(t)\,, \quad 0 \le t < \infty\,. \tag{1.5}$$

We shall call (1.5) the Itô-McKean representation of the Feller diffusion. With this representation Th. Brox (in [3]) studied the limit of the scaled process $\alpha^{-2}X(e^{\alpha})$ (and the limit of the form $\alpha^{-2}X(e^{\alpha h(\alpha)})$, where $h(\alpha) \to 1$) as $\alpha \to \infty$.

After this work of Brox ([3]) there have been a number of papers devoted to the study of the process $X(t)$ defined by (1.5). Let us only mention the papers [1,4,20] where the local time of $X(t)$ is studied. Some ideas in these papers will be used later. Let us also mention that about the same time as [3] the process $X(t)$ was also studied in the paper [19].

It may be interesting to note that if W were continuously differentiable, it could be easily checked by Itô's calculus that such an X defined by (1.5) is a weak solution to (1.1) (see Remark 3.3 (i) in Section 3).

By definition a diffusion is a Markov process with continuous sample paths. Probabilists are interested in more detailed properties of the sample paths. By fixing an almost sure realization of two-sided Brownian motion W, the equation (1.1) can be considered as a stochastic differential equation with singular drift in the form

$$X_t = X_0 + \int_0^t \sigma(X_s)d\mathcal{B}(s) + \int_0^t b'(X_s)ds, \tag{1.6}$$

where $\mathcal{B}$ is a Brownian motion, and σ and b are continuous function. In fact, there have been already a number of work on such (one dimensional) equations (see e.g. [2], [5], [6], [18], and the references therein). In some cases strong existence and uniqueness has been proved for such equations. In the case $\sigma \equiv 1$ (which, in fact, is the situation in (1.1)) if b is Hölder continuous of order α for some $\alpha > 1/2$, then the existence and uniqueness of the strong solution to (1.6) were derived in [2]. Under similar conditions, these results have been also proved in [18]. However, it seems that in the case of the function b being less regular than Hölder of order $1/2$, the representation for X which is known is via solution of certain martingale problem, or time change analogous to (1.5) or via weak solution to (1.6), where the last term on the right hand side of the equation is defined as an extension of a certain map (see e.g. Corollary 3.4 and Remark 3.5 in [5] or Corollary 5.13 and Remark 5.14 in [18].) We would like to mention that existence and uniqueness of the strong solution to (1.6) has been also obtained in [18] under some technical assumption $\mathcal{A}(\nu_0)$ (see [18, pg. 2229]). It is not clear whether this technical assumption can be verified for the equation (1.1) which corresponds to (1.6) with $\sigma = 1$ and $b' = -\frac{1}{2}\dot{W}$.

The current paper offers the following contributions: First, we show that for any Brownian motion B, independent of W, the Itô-McKean representation (1.5) is a weak solution of the equation (1.1); second, for any given Brownian motion $\mathcal{B}$ we construct a particular Brownian motion B, independent of W, such that the Itô-McKean representation (1.5) is a strong solution of the equation (1.1); third, we show the strong uniqueness of the solution; and finally, we develop an Itô calculus for the solution. Note that the regularity of the generalized drift $b' = -\frac{1}{2}\dot{W}$ (where W is Hölder continuous with exponent α, for any α less than $1/2$) is at the border of what the papers mentioned above handled to show that X is a solution of the stochastic differential equation with generalized drift. While proving our results, a major task for us is to give a meaning to the integral $\int_0^t \dot{W}(X(s))\,ds$ appearing in (1.1) and its approximations. We shall complete this task by using the local time of a Brownian motion and the following identity:

$$\int_0^t \dot{W}(X(s))ds = \int_{\mathbb{R}} e^{-W(x)} L_B(\xi, S_W(x)) W(dx)\Big|_{\xi = T^{-1}_{W,B}(t)}.$$

[See (2.11) in the next section.] However, due to the lack of martingale property of $L_B(\xi, y)$ on ξ, we need to use Garsia-Rodemich-Rumsey theorem in order to give a meaning to the above object. This in turn forces us to study the higher moment properties of the local time of Brownian motion, which has its own interest. Let us also point out that our approach is probabilistic and we crucially use the fact that W is a Brownian motion. In comparison with the results obtained in the aforementioned papers, the other results can be applied to (almost) every sample path of W, but need to assume that W has a Hölder continuity higher than $1/2$, which cannot be verified by a Brownian motion. Our result can be applied to Brownian motion but is not for every sample path.

Notations: Throughout the paper we will use a number of different filtrations and σ-fields. Set $\mathcal{F}^B = \{\mathcal{F}^B_t\}_{t\geq 0}$ be the filtration generated by the Brownian motion B. We will also need the extended filtration $\mathcal{F}^{B,W} = \{\mathcal{F}^{B,W}_t\}_{t\geq 0}$ given by

$$\mathcal{F}^{B,W}_t = \mathcal{F}^B_t \bigvee \sigma(W(x), x \in \mathbb{R}),\ t \geq 0.$$

$C_b(\mathbb{R})$ denotes the space of all bounded continuous functions on $\mathbb{R}$. For $\lambda \in (0,1)$, and $a < b$, let $\|\cdot\|_{\lambda,[a,b]}$ the λ-Hölder norm for functions on $[a,b]$, that is,

$$\|f\|_{\lambda,[a,b]} \equiv \|f\|_{\infty,[a,b]} + \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^{\lambda}} \tag{1.7}$$

where $\|\cdot\|_{\infty,[a,b]}$ is the supremum norm. Similarly $\|\cdot\|_{\lambda}$ will denote the λ-Hölder norm for functions on $\mathbb{R}$. Let $C^{\lambda}([a,b])$ (resp. C^{λ}) be the space of Hölder continuous functions f on $[a,b]$ (resp. on $\mathbb{R}$) with $\|f\|_{\lambda,[a,b]} < \infty$ (resp. $\|f\|_{\lambda,\mathbb{R}} < \infty$). The notation $A \lesssim B$ means $A \le CB$ for some non-negative constant C.

2. Main results

It is evident that to understand equation (1.1), one should first properly define the drift term $\int_0^t \dot{W}(X(s))ds$. For a two-sided Brownian motion W, $\dot{W}$ is not a function but a distribution (generalized functions), this integral has no canonical meaning. However, if the process X admits the Itô-McKean presentation (1.5) for some Brownian motion B independent of W, we can define this integral in such a way that the map $W \mapsto \int \dot{W}(X(s))ds$ is an extension of the integration on smooth functions, i.e $\int \dot{f}(X(s))ds$ for a regular function f.

Let us now describe our method in more details by the following heuristic argument. We first fix W and B, and adopt the following strategy. Let $L_X(t,x)$ be the local time of the process X which is defined as the unique process such that

$$\int_0^t f(X(s))ds = \int_{\mathbb{R}} L_X(t,x)f(x)dx\,, \quad \forall\, t \ge 0 \quad \text{and} \quad \forall f \in C_b(\mathbb{R}). \tag{2.8}$$

From the representation (1.5), we see that

$$L_X(t,x) = e^{-W(x)} L_B(T^{-1}_{W,B}(t), S_W(x))\,, \tag{2.9}$$

where $L_B(t,x)$ is the local time for Brownian motion B, S_W and $T_{W,B}$ are defined by (1.3) and (1.4). Using the definition (2.8) of the local time, we formally write

$$\int_0^t \dot{W}(X(s))ds = \int_{\mathbb{R}} L_X(t,x)\dot{W}(x)dx = \int_{\mathbb{R}} L_X(t,x)W(d^o x)\,. \tag{2.10}$$

A fundamental problem arises: in what sense should one interpret $W(d^o x)$, the above stochastic integral with respect to W? Note that for fixed t, the process $x \mapsto L_X(t,x)$ is not necessarily adapted, which is one of the difficulties. If W were a smooth function the above integral would be the usual (pathwise) integral. Hence the last integral in (2.10) should be defined as the (anticipative) Stratonovich stochastic integral so that the integrations in (2.10) are extensions of the classical setting of smooth functions. It turns out that with this interpretation, the process X given by (1.5) will indeed solve (1.1) (weakly). This can also been seen from our approximation argument described in Section 3.

Let us explain how the Stratonovich integral $\int_{\mathbb{R}} L_X(t,x)W(d^o x)$ can be defined rigorously. Presumably, one may use the anticipative stochastic calculus ([16]) (with the help of Malliavin calculus) to define this integral. However, we immediately encountered a difficulty to show the square integrability of $L_X(t,x)$. Instead, we

use (2.9) and (2.10) to formally write

$$\int_0^t \dot{W}(X(s))ds = \int_{\mathbb{R}} L_X(t,x)W(d^o x) = \int_{\mathbb{R}} e^{-W(x)} L_B(T_{W,B}^{-1}(t), S_W(x))W(d^o x)$$

$$= \int_{\mathbb{R}} e^{-W(x)} L_B(\xi, S_W(x))W(d^o x)\Big|_{\xi=T_{W,B}^{-1}(t)}. \tag{2.11}$$

The precise definition of the expression on the right hand side of (2.11) will be given in this section, and eventually this will enable us to give a meaning to $\int_0^t \dot{W}(X(s))ds$ (see Definition 2.4 below).

In fact, throughout the paper, we can consider a more general situation, namely the integral of the type

$$\int_0^t g(X(s), W(X(s)))\dot{W}(X(s))ds. \tag{2.12}$$

This generalization will later allow us to develop Itô calculus on equation (1.1) and obtain strong uniqueness result. Concerning the function g, we assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a *deterministic* continuous function such that

- For every $x \in \mathbb{R}$, the function $u \mapsto g(x,u)$ is continuously differentiable,
- For every $u \in \mathbb{R}$, the functions $x \mapsto g(x,u)$ and $x \mapsto \partial_u g(x,u)$ are Hölder continuous of order λ with $\lambda > 1/2$.

In addition, we assume that g satisfies the analytic bounds

$$\sup_{x\in K} |g(x,u)| \le c_1(K)e^{\theta|u|} \tag{2.13}$$

and

$$\sup_{x,y\in K} \frac{|g(x,u)-g(y,u)|}{|x-y|^\lambda} + \sup_{x,y\in K} \frac{|\partial_u g(x,u)-\partial_u g(y,u)|}{|x-y|^\lambda} \le c_2(K)e^{\theta|u|} \tag{2.14}$$

for every $u \in \mathbb{R}$ and compact interval K, where $\theta, c_1(K)$ and $c_2(K)$ are some positive constants.

Note that for any fixed $\xi \ge 0$, the mapping $x \mapsto g(x, W(x))L_B(\xi, S_W(x)), x \in \mathbb{R}_+$ is *adapted* with respect to the filtration generated by $\{W(z),\ z \in [0,x]\}_{x\ge 0}$. Similarly the mapping $x \mapsto g(x, W(x))L_B(\xi, S_W(x)), x \in \mathbb{R}_-$ is *adapted* with respect to the filtration generated by $\{W(z),\ z \in [x,0]\}_{x\le 0}$. To elaborate this point, we define

$$\widetilde{W}(x) = W(-x),\ x \ge 0.$$

Let $W(dx)$ and $\widetilde{W}(dx)$ denote Itô differentials. Then for any $a \le b$, and continuous function g on $\mathbb{R}^2$, we define the Itô integral

$$\int_a^b g(x,W(x))L_B(\xi,S_W(x))W(dx) \tag{2.15}$$

$$= \begin{cases} \int_a^b g(x,W(x))L_B(\xi,S_W(x))W(dx), \text{ if } 0 \le a \le b \\[2ex] \int_0^{|a|} g(x,W(-x))L_B(\xi,S_W(-x))\widetilde{W}(dx) \\ \qquad + \int_0^b g(x,W(x))L_B(\xi,S_W(x))W(dx), \text{ if } a \le 0 \le b, \\[2ex] \int_{|b|}^{|a|} g(x,W(-x))L_B(\xi,S_W(-x))\widetilde{W}(dx), \text{ if } a \le b \le 0. \end{cases}$$

Now for any $a \le b$, $\xi > 0$, and any continuous function g satisfying (2.13) and (2.14), we define

(2.16)
$$\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x) := \int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(dx) - \frac{1}{2}\int_a^b \partial_u g(x, W(x)) L_B(\xi, S_W(x)) dx\,,$$

where $\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(dx)$ is the Itô stochastic integral defined in (2.15). While the right hand side of (2.16) is valid for bigger classes of functions, we restricted ourselves to conditions (2.13) and (2.14) because it is this specific class in which most of the limiting results of the current work hold. The following result, whose proof is given in Section 6, confirms that the integration defined in (2.16) is indeed of Stratonovich type.

Proposition 2.1. *Assume that g satisfies the conditions (2.13) and (2.14) with some $\lambda > 1/2$. In addition, we assume that $u \mapsto \partial_u g(x, u)$ is continuously differentiable. Fix arbitrary $a < b$. Let $\pi : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a, b]$ and let $|\pi| = \max_{0 \le i \le n-1}(x_{i+1} - x_i)$. Let*

$$W_\pi(x) = W(x_i) + (W(x_{i+1}) - W(x_i)) \frac{x - x_i}{x_{i+1} - x_i}\,, \quad x_i \le x < x_{i+1}\,, \tag{2.17}$$

be the linear interpolation of W associated with the partition π. Then

$$\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x) = \lim_{|\pi| \to 0} \int_a^b g(x, W(x)) L_B(\xi, S_W(x)) \dot{W}_\pi(x) dx\,, \tag{2.18}$$

where the limit in (2.18) is in L^2.

The regularity of this integration is described in the following result.

Lemma 2.2. *Let g be a continuous function satisfying (2.13) and (2.14). Then there exists a version of the process*

$$(\xi, a) \mapsto \int_{-a}^a g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x)$$

which is jointly continuous in $(\xi, a) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. From (2.16), it is sufficient to show the process

$$H(\xi, y) = \int_0^y g(x, W(x)) L_B(\xi, S_W(x)) dW(x)$$

has a jointly continuous version. Fix $y_1 < y_2 < N$, $\xi_1 < \xi_2$, using martingale moment inequality and (2.13), we obtain

$$\begin{aligned}&\mathbb{E}|H(\xi_1,y_1)-H(\xi_1,y_2)-H(\xi_2,y_1)+H(\xi_2,y_2)|^4\\&=\mathbb{E}\left|\int_{y_1}^{y_2} g(x,W(x))L_B([\xi_1,\xi_2],S_W(x))dW(x)\right|^4\\&\lesssim |y_2-y_1|\int_{y_1}^{y_2}\mathbb{E}e^{4\theta|W(x)|}|L_B([\xi_1,\xi_2],S_W(x))|^4dx\,.\end{aligned}$$

It is straightforward to verify that (see also the identity (8.1) below)

$$\mathbb{E}^B|L_B([\xi_1,\xi_2],S_W(x))|^4\le C|\xi_2-\xi_1|^2\,.$$

Hence,

$$\begin{aligned}&\mathbb{E}|H(\xi_1,y_1)-H(\xi_1,y_2)-H(\xi_2,y_1)+H(\xi_2,y_2)|^4\\&\qquad\le C|y_2-y_1||\xi_2-\xi_1|^2\int_{y_1}^{y_2}e^{8\theta^2|x|}dx\le C_N|y_2-y_1|^2|\xi_2-\xi_1|^2\,.\end{aligned}$$

The result then follows from two-parameter Kolmogorov theorem. □

As an immediate consequence, we have

Lemma 2.3. *Let g be a continuous function satisfying (2.13) and (2.14). Then for any fixed $\xi \ge 0$, the limit*

$$\lim_{a\to\infty}\int_{-a}^{a} g(x,W(x))L_B(\xi,S_W(x))W(d^ox)$$

exists almost surely. We will denote the limiting process as

$$\int_{-\infty}^{\infty} g(x,W(x))L_B(\xi,S_W(x))W(d^ox)\,.$$

Furthermore, for any fixed $\xi \ge 0$, we define

$$\tau_{W,B}(\xi)=\inf\{x>0:S_W(x)>|\max_{s\in[0,\xi]}B_s|\}\,. \tag{2.19}$$

Then,

$$\tau_{W,B}(\xi)<\infty,\ \text{a.s.,} \tag{2.20}$$

and for all $\xi \ge 0$,

$$\begin{aligned}&\int_{-\infty}^{\infty} g(x,W(x))L_B(\xi,S_W(x))W(d^ox)\\&\qquad=\int_{-\tau_{W,B}(\xi)}^{\tau_{W,B}(\xi)} g(x,W(x))L_B(\xi,S_W(x))W(d^ox)\,.\end{aligned} \tag{2.21}$$

As a consequence, the process $\xi\mapsto\int_{-\infty}^{\infty} g(x,W(x))L_B(\xi,S_W(x))W(d^ox)$ has a continuous version.

Proof. We denote $M_B(\xi)=|\max_{s\in[0,\xi]}B_s|$. A result of Matsumoto and Yor in [15, identity (4.5)] shows that

$$\lim_{K\to\infty}\sqrt{2\pi K}\,\mathbb{E}[S_W(K)^{-1}]=1\,. \tag{2.22}$$

On the other hand, for each $K>0$ (recall also that B and W are independent)

$$\begin{aligned}P(\tau_{W,B}(\xi)>K)&=P(S_W(K)^{-1}\geq M(\xi)^{-1})\\&\leq \mathbb{E}[M_B(\xi)]\mathbb{E}[S_W(K)^{-1}]\lesssim \mathbb{E}[S_W(K)^{-1}]\,.\end{aligned}$$

Together with (2.22), it follows that $\lim_{K\to\infty}P(\tau_{W,B}(\xi)>K)=0$. From here, we deduce (2.20).

Since $S_W(\cdot)$ is strictly increasing, if y is such that $y>\tau_{W,B}(\xi)$, then $S_W(y)>|\max_{s\in[0,\xi]}B_s|$, and hence $L_B(\xi,S_W(y))$ vanishes. As a consequence, with probability one, the map $x\mapsto g(x,W(x))L_B(\xi,S_W(x))$ is supported in the interval $[-\tau_{W,B}(\xi),\tau_{W,B}(\xi)]$. Therefore, the limit of $\int_{-a}^a g(x,W(x))L_B(\xi,S_W(x))W(d^ox)$ as a goes to ∞ exists almost surely. From here, we also obtain (2.21). By Lemma 2.2, the map $(\xi,a)\mapsto\int_{-a}^a g(x,W(x))L_B(\xi,S_W(x))W(d^ox)$ is continuous. This together with continuity of $\xi\mapsto\tau_{W,B}(\xi)$ implies that the process

$$\xi\mapsto\int_{-\infty}^\infty g(x,W(x))L_B(\xi,S_W(x))W(d^ox)$$

has a continuous version. □

With the help of Lemmas 2.2, 2.3 we can now define the integral of the type (2.12) for sufficiently regular functions g and X as in (1.5).

Definition 2.4. Let X be the process in (1.5). Suppose that g is a function satisfying conditions (2.13) and (2.14). Then for every $t\geq0$, we define

$$\begin{aligned}&\int_0^t g(X(s),W(X(s)))\dot W(X(s))ds\\&\quad:=\int_{-\infty}^\infty g(x,W(x))e^{-W(x)}L_B(\xi,S_W(x))W(d^ox)\big|_{\xi=T^{-1}_{W,B}(t)},\end{aligned}\tag{2.23}$$

where $T_{W,B}$ is defined by (1.4) and $T^{-1}_{W,B}$ is the inverse of $T_{W,B}$. In particular, for $g\equiv1$ we have

$$\int_0^t\dot W(X(s))ds=\int_{-\infty}^\infty e^{-W(x)}L_B(\xi,S_W(x))W(d^ox)\big|_{\xi=T^{-1}_{W,B}(t)}\tag{2.24}$$

for all $t\geq0$.

From Lemma 2.3, the process $\xi\mapsto\int_{-\infty}^\infty g(x,W(x))e^{-W(x)}L_B(\xi,S_W(x))W(d^ox)$ has a continuous version. In addition, since the map $t\mapsto T^{-1}_{W,B}(t)$ is also continuous, we see that the process

$$t\mapsto\int_0^t g(X(s),W(X(s)))\dot W(X(s))ds$$

also has a continuous version. From now on, we will only consider this continuous version whenever we write either $\int_0^t g(X(s),W(X(s)))\dot W(X(s))ds$ or alternatively its two other equivalent presentations

$$\begin{aligned}&\int_{-\infty}^\infty g(x,W(x))e^{-W(x)}L_B(T^{-1}_{W,B}(t),S_W(x))W(d^ox)\\&\qquad=\int_{-\infty}^\infty g(x,W(x))L_X(t,x)W(d^ox)\,.\end{aligned}$$

In the above, the equality can be seen from (2.9).

Now with a rigorous definition of $\int_0^t \dot{W}(X(s))ds$ at hand we can now precisely describe the notions of strong and weak solutions to (1.1).

Definition 2.5 (Strong solution)**.** Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion, and $(\mathcal{B}(t), t \geq 0)$ be a Brownian motion with respect to a usual filtration $(\mathcal{F}^{\mathcal{B}})_{t\geq 0}$, independent of W. Let $\mathcal{F}^{\mathcal{B},W} = (\mathcal{F}_t^{\mathcal{B},W})_{t\geq 0}$ be the extended filtration given by

$$\mathcal{F}_t^{\mathcal{B},W} = \mathcal{F}_t^{\mathcal{B}} \bigvee \sigma(W(x), x \in \mathbb{R}) \,, \forall t \geq 0 \,.$$

We assume that $\mathcal{F}^{\mathcal{B},W}$ also satisfies the usual conditions. A continuous process $(X(t), t \geq 0)$ is a strong solution to (1.1) if it satisfies the following conditions:

(i) X is adapted to the extended filtration $\mathcal{F}^{\mathcal{B},W}$.

(ii) There exists a Brownian motion $(B(t), t \geq 0)$ independent of W such that $X(t)$ admits the Itô-McKean representation (1.5).

(iii) For every t, the integral $\int_0^t \dot{W}(X(s))ds$ is well defined as in Definition 2.4.

(iv) For every $t \geq 0$, the equation

$$X(t) = \mathcal{B}(t) - \frac{1}{2}\int_0^t \dot{W}(X(s))ds$$

holds almost surely.

Definition 2.6 (Weak solution)**.** Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. A pair $(X, \mathcal{B})$ is a weak solution to (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ if it satisfies the following conditions:

(i) X is a continuous process process adapted to $(\mathcal{F}_t)_{t\geq 0}$ and $\mathcal{B}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion independent of W.

(ii) There exists a Brownian motion $(B(t), t \geq 0)$ independent of W such that $X(t)$ admits the Itô-McKean representation (1.5).

(iii) For every $t \geq 0$, the integral $\int_0^t \dot{W}(X(s))ds$ is well defined as in Definition 2.4.

(iv) For every $t \geq 0$, the equation

$$X(t) = \mathcal{B}(t) - \frac{1}{2}\int_0^t \dot{W}(X(s))ds$$

holds almost surely.

The major contribution of the current paper is the strong existence and uniqueness result for the Brox equation (1.1).

Theorem 2.7 (Existence and uniqueness of strong solution)**.** *Let W be a two-sided Brownian motion and $\mathcal{B}$ be a Brownian motion independent of W. Then there exists a unique strong solution X to (1.1).*

In proving Theorem 2.7, we are able to obtain existence of a pair $(X, \mathcal{B})$ satisfying (1.1). The precise statement is following.

Proposition 2.8 (Existence of a weak solution)**.** *Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion and let $(B(t), t \geq 0)$ be a Brownian motion, independent of W. Let $X(t)$ be the Itô-McKean representation given by the equation (1.5) and let $\int_0^t \dot{W}(X(s))ds$ be defined by (2.24). Then, there is a Brownian motion $\mathcal{B}$ determined by*

$$\mathcal{B}(t) = \int_0^t e^{-W\circ S_W^{-1}\circ B\circ T_{W,B}^{-1}(s)} dB \circ T_{W,B}^{-1}(s) \,, \tag{2.25}$$

which is independent of W, such $(X,\mathcal{B})$ is a weak solution to equation (1.1).

In fact, Theorem 2.8 claims a bit more than just weak existence. It states that any Brox diffusion given by the Itô-McKean representation (1.5) is a weak solution to the equation (1.1). In addition, the Brownian motion $\mathcal{B}$ appeared in the equation is given explicitly by the equation (2.25).

As an application of our method, we can easily obtain the following Itô formula whose proof is provided in Section 4.

Theorem 2.9 (Itô formula)**.** *Let $(X,\mathcal{B})$ be a weak solution to (1.1). Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a deterministic continuous function such that*

- *For every x, the map $u\mapsto f(x,u)$ is continuously differentiable*
- *f and $\partial_u f$ satisfy the conditions (2.13) and (2.14).*

We define the function $F(x)=\int_0^x f(y,W(y))dy+F(0)$, where $F(0)$ is some constant. Then, with probability one,

$$F(X(t))=F(0)+\int_0^t f(X(s),W(X(s))d\mathcal{B}(s)+\frac12\int_0^t \partial_x f(X(s),W(X(s)))ds$$
$$-\frac12\int_{-\infty}^{\infty} f(x,W(x))L_X(t,x)W(d^ox)+\frac12\int_{-\infty}^{\infty}\partial_u f(x,W(x))L_X(t,x)W(d^ox)\,.$$

An immediate corollary is the following

Corollary 2.10 (Itô formula)**.** *Let $(X,\mathcal{B})$ be a weak solution to (1.1). Let $F:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a measurable deterministic function which is continuously differentiable in t and twice continuously differentiable in x. Then, with probability one,*

$$F(t,X(t))=F(0,0)+\int_0^t \partial_s F(s,X(s))ds+\int_0^t \partial_x F(s,X(s))d\mathcal{B}(s)$$
$$+\frac12\int_0^t \partial_{xx}F(s,X(s))ds-\frac12\int_{-\infty}^{\infty}\partial_x F(s,x)L_X(t,x)W(d^ox)\,.$$

The rest of this paper is organized as follows. In the next section, we provide some preliminaries and show how Proposition 2.8 can be derived. Theorem 2.9 is proved in Section 4. The proof of Theorem 2.7 is given in Section 5. The proof of Proposition 2.1 is provided in Section 6. Proofs of some further technical results (described in Section 3) are provided in Sections 7, 8 and 9.

3. Preliminaries and proof of Proposition 2.8

We present in the current section some necessary results which will be used several times throughout our paper. Since Proposition 2.8 follows directly from these results, we provide its proof at the end of the section.

Let $\widetilde{\mathcal{F}}=\{\widetilde{\mathcal{F}}_t\}_{t\ge0}$ be a filtration under which B is a Brownian motion. We assume that the filtration $\widetilde{\mathcal{F}}$ satisfies the usual conditions for a filtration; namely, it is right-continuous and $\widetilde{\mathcal{F}}_0$ contains all the null sets. In what follows, $\widetilde{\mathcal{F}}$ is usually chosen to be $\mathcal{F}^{B,W}$.

An $\{\widetilde{\mathcal{F}}_t\}$*-time-change* is a càdlàg, increasing family of $\{\widetilde{\mathcal{F}}_t\}$-stopping times. It is said to be *finite* if each stopping time is finite almost surely, and *continuous* if it is almost surely continuous with respect to time. Let $T=\{T(t):t\ge0\}$ be

a finite $\{\widetilde{\mathcal{F}}_t\}$-time change and consider the time-changed filtration $\{\widetilde{\mathcal{F}}_{T_t}\}_{t\geq 0}$. The right-continuity of $\{\widetilde{\mathcal{F}}_t\}$ and $\{T_t\}$ imply that $\{\widetilde{\mathcal{F}}_{T_t}\}_{t\geq 0}$ satisfies the usual conditions. Moreover, the time-changed process $\{B\circ T(t)\}$ is an $\{\widetilde{\mathcal{F}}_{T_t}\}$-semimartingale (see [12, Corollary 10.12]). As a consequence, one can define the Itô integral of the form $\int_0^t g(B\circ T(s))dB\circ T(s)$. In the following proposition we gather some useful facts.

Proposition 3.1. *Let f be a function in $C^2(\mathbb{R})$, the set of continuous functions with continuous derivatives up to second order. Let $T=(T(t),t\geq 0)$ be a continuous finite time change. Then, with probability one, for all $t\geq 0$, the following identities hold*

$$f(B\circ T(t))=f(B\circ T(0))+\int_{T(0)}^{T(t)}f'(B(s))dB(s)+\frac{1}{2}\int_{T(0)}^{T(t)}f''(B(s))ds\,, \tag{3.26}$$

$$\int_{T(0)}^{T(t)}f'(B(u))dB(u)=\int_0^t f'(B\circ T(s))dB\circ T(s)\,, \tag{3.27}$$

$$\int_{T(0)}^{T(t)}f''(B(u))du=\int_0^t f''(B\circ T(s))dT(s)\,. \tag{3.28}$$

Finally, the process $t\mapsto\int_0^t f'(B\circ T(s))dB\circ T(s)$ is a semimartingale with respect to the filtration $\{\widetilde{\mathcal{F}}_{T_t}\}_{t\geq 0}$,and its quadratic variation is given by

$$\langle\int_0^t f'(B\circ T(s))dB\circ T(s)\rangle=\int_0^t|f'(B\circ T(s))|^2dT(s)\,. \tag{3.29}$$

In fact, in [13], the author has obtained time-changed Itô formula (such as (3.26)) for semimartingales possibly with jumps. However, we do not need such general result in the current paper. We refer the reader to [13, Theorem 3.3] for a justification of (3.26) and (3.29). Identities (3.27) and (3.28) follow from [12, Proposition 10.21], see also in [13].

Throughout the paper, we will approximate $W(x)$ by its polygonal approximations. Since $W(x)$ is defined for all $x\in\mathbb{R}$, we now partition the whole line $\mathbb{R}$. Let π be any partition with nodes $\{x_i\in\mathbb{R}:x_i<x_{i+1}\ \ \forall i\in\mathbb{Z}\}$. Then the polygonal approximation of W associated with this partition, denoted by W_π, is the piecewise function such that for every $i\in\mathbb{Z}$

$$W_\pi(x)=W(x_i)+\frac{W(x_{i+1})-W(x_i)}{x_{i+1}-x_i}(x-x_i)\ ,\quad x_i\leq x<x_{i+1}\,. \tag{3.30}$$

Fix arbitrary Brownian motion B independent of W. Then, for any polygonal approximation W_π of W, we can define X_π via an analogue to the Itô-McKean representation (1.5):

$$X_\pi(t)=S_{W_\pi}^{-1}\circ B\circ T_{W_\pi,B}^{-1}(t)\,,\quad 0\leq t<\infty\,, \tag{3.31}$$

where

$$S_{W_\pi}(x)=\int_0^x e^{W_\pi(z)}dz,\ \ ,\quad 0\leq t<\infty, \tag{3.32}$$

and

$$T_{W_\pi,B}(t)=\int_0^t e^{-2W_\pi\circ S_{W_\pi}^{-1}(B(s))}ds\,,\quad 0\leq t<\infty. \tag{3.33}$$

We also denote

$$\mathcal{B}_\pi(t)=\int_0^t e^{-W_\pi(X_\pi(s))}dB\circ T^{-1}_{W_\pi,B}(s)\,, \tag{3.34}$$

Since W_π is piecewise differentiable it follows from Proposition 3.1 that

Lemma 3.2. *Let $X_\pi(t)$ be defined by (3.31)-(3.33) and $\mathcal{B}_\pi(t)$ be defined in (3.34). Then $\mathcal{B}_\pi$ is a Brownian motion with respect to the time-changed filtration $\{\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(t)}\}_{t\ge0}$. In addition, $\mathcal{B}_\pi$ is independent of W and X_π satisfies*

$$X_\pi(t)=-\frac12\int_0^t \dot W_\pi(X_\pi(s))ds+\mathcal{B}_\pi(t)\,. \tag{3.35}$$

Proof. The Itô formulas (3.26)-(3.28) from Proposition 3.1 give

$$\begin{aligned}X_\pi(t)=&\int_0^t (S^{-1}_{W_\pi})'\circ B\circ T^{-1}_{W_\pi,B}(s)dB\circ T^{-1}_{W_\pi,B}(s)\\&+\frac12\int_0^t (S^{-1}_{W_\pi})''\circ B\circ T^{-1}_{W_\pi,B}(s)\frac{d}{ds}T^{-1}_{W_\pi,B}(s)ds\,.\end{aligned} \tag{3.36}$$

Note that we apply Proposition 3.1 by, first, fixing a realization of W; we also use the fact that B is a Brownian motion with respect to $\mathcal{F}^{B,W}$. From the definition of $S_{W_\pi}(x)$, we have

$$\frac{d}{dx}S^{-1}_{W_\pi}(x)=e^{-W_\pi(S^{-1}_{W_\pi}(x))}\,,$$

$$\frac{d^2}{dx^2}S^{-1}_{W_\pi}(x)=-e^{-2W_\pi(S^{-1}_{W_\pi}(x))}\dot W_\pi(S^{-1}_{W_\pi}(x))\,.$$

Thus

$$\left[\frac{d}{dx}S^{-1}_{W_\pi}\right]\circ B\circ T^{-1}_{W_\pi,B}(s)=e^{-W_\pi(X_\pi(s))}\,,$$

$$\left[\frac{d^2}{dx^2}S^{-1}_{W_\pi}\right]\circ B\circ T^{-1}_{W_\pi,B}(s)=-e^{-2W_\pi(X_\pi(t))}\dot W_\pi(X_\pi(s))\,.$$

Similarly, we have

$$\frac{d}{dt}T^{-1}_{W_\pi,B}(s)=e^{2W_\pi(S^{-1}_{W_\pi}\circ B\circ T^{-1}_{W_\pi,B}(s))}=e^{2W_\pi(X_\pi(s))}\,.$$

Thus (3.36) can be written as

$$\begin{aligned}X_\pi(t)\;=\;&\int_0^t e^{-W_\pi(X_\pi(s))}dB\circ T^{-1}_{W_\pi,B}(s)\\&+\frac12\int_0^t -e^{-2W_\pi(X_\pi(t))}\dot W_\pi(X_\pi(s))e^{2W_\pi(X_\pi(s)}ds\\=\;&\mathcal{B}_\pi(t)-\frac12\int_0^t \dot W_\pi(X_\pi(s))ds\,.\end{aligned} \tag{3.37}$$

From Doob's optional stopping (sampling) theorem it is easy to see that $(\mathcal{B}_\pi(t),t\ge 0)$ is a local martingale with respect to $\{\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(t)}\}_{t\ge0}$. Moreover, its quadratic variation is

$$\int_0^t e^{-2W_\pi(X_\pi(s))}\frac{d}{ds}T^{-1}_{W_\pi,B}(s)ds=\int_0^t e^{-2W_\pi(X_\pi(s))}e^{2W_\pi(X_\pi(s))}ds=t\,. \tag{3.38}$$

Thus by Lévy's characterization theorem $\mathcal{B}_\pi(t)$ is a Brownian motion with respect to $\{\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(t)}\}_{t\geq 0}$.

To complete the proof of Lemma 3.2, it remains to show that $\mathcal{B}_\pi$ and W are independent processes. Since both of them are Gaussian, it suffices to show that they are uncorrelated. Indeed, using (3.27) and (3.31), we can write

$$\mathcal{B}_\pi(t) = \int_0^{T^{-1}_{W_\pi,B}(t)} e^{-W_\pi\circ S^{-1}_{W_\pi}\circ B(u)} dB(u)\,.$$

Hence, for every $t \geq 0$ and $x \in \mathbb{R}$, we use the fact that $\mathcal{B}_\pi$ is $\{\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(t)}\}_{t\geq 0}$-Brownian motion, and the fact that W is measurable with respect to $\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(0)}$ to get

$$\mathbb{E}\left[\mathcal{B}_\pi(t)W(x)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{B}_\pi(t)\,\middle|\,\mathcal{F}^{B,W}_{T^{-1}_{W_\pi,B}(0)}\right]W(x)\right] = \mathbb{E}\left[\mathcal{B}_\pi(0)W(x)\right] = 0.$$

Hence, we complete the proof of Lemma 3.2. □

Remark 3.3. (i) Lemma 3.2 implies that $(X_\pi(t), t \geq 0)$ is the weak solution of the equation:

$$dX_\pi(t) = -\frac{1}{2}\dot{W}_\pi(X_\pi(t))dt + d\widetilde{B}(t),$$

where $\widetilde{B}$ is a Brownian motion independent of W_π.

(ii) The result of Lemma 3.2 holds true when $W_\pi(x)$ is replaced by any continuously differentiable function.

Now Proposition 2.8 follows from (3.35) by shrinking the mesh size $|\pi|$ to 0. This step is verified through the following propositions.

Proposition 3.4. *For every $T \geq 0$, $\lim_{|\pi|\to 0}\mathbb{E}\sup_{t\leq T}|\mathcal{B}_\pi(t) - \mathcal{B}(t)|^2 = 0$.*

Proposition 3.5. *Then for every $\delta > 0$, there exists a partition $\pi(\delta)$ of $\mathbb{R}$ such that for any $T > 0$,*

$$\lim_{\delta\to 0}\sup_{t\leq T}\left|\int_0^t g(X_{\pi(\delta)}(s), W_{\pi(\delta)}\circ X_{\pi(\delta)}(s))\dot{W}_{\pi(\delta)}(X_{\pi(\delta)}(s))ds \right.$$
$$\left. - \int_{-\infty}^{\infty} g(x, W(x))L_X(t,x)W(d^o x)\right| = 0,$$

with probability one.

The proofs of the above two propositions are provided in Section 7 and Section 9 respectively. Proposition 3.5 in turn is relied on the following moment estimates for local time of Brownian motion, which are of independent interest.

Proposition 3.6. *(i) Let $x, y \in \mathbb{R}$. For every $\beta \in [0, 1/2]$, the following estimates holds*

$$\left|\mathbb{E}\left(L_B([\xi,\eta],y) - L_B([\xi,\eta],x)\right)^{2n}\right| \leq C_{\beta,n}|\eta - \xi|^{n(1-\beta)}|x-y|^{2\beta n}\,. \tag{3.39}$$

(ii) For every $x_1, y_1, \cdots, x_k, y_k$ satisfying

$$x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_{2n} < y_{2n}\,. \tag{3.40}$$

and every $\alpha \in [0,1]$ *we have*

$$\left|\mathbb{E}\prod_{k=1}^{2n}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\right| \le C_{\alpha,n}|\eta-\xi|^{n\alpha}\prod_{k=1}^{2n}|y_k-x_k|^{1-\alpha}\,. \tag{3.41}$$

The proof of the previous proposition is given in Section 8.

Remark 3.7. (i) The former inequality (3.39) is well known. The above second estimate (3.41) is new and quite interesting itself. Since in our proof of (3.41) we shall obtain some results which can be used to prove (3.39) easily, we shall also present a straightforward proof of (3.39).

(ii) From [17], it is known that $L(\xi,x)$ is a semimartingale on x. A consequence is that $\mathbb{E}\left(L_B(\xi,y)-L_B(\xi,x)\right)^{2n} \le C_{\beta,n}|x-y|^n$. (3.39) is an extension of this inequality.

We will also need the following analytic result.

Lemma 3.8. *Let* f *and* f_n*,* $(n=1,2,\ldots)$ *be bijective functions on* $\mathbb{R}$ *which are continuous and strictly increasing. Suppose that* $f_n(x)$ *converges to* $f(x)$ *for every* x *in* $\mathbb{R}$*. Then for any compact* $A\subset\mathbb{R}$*,* $\lim_{n\to\infty}\sup_{y\in A}|f_n^{-1}(y)-f^{-1}(y)|=0$.

Proof. The proof follows by contradiction. Suppose there exists ϵ_0 and a subsequences $\{f_{n_k}\}$ and $\{y_{n_k}\}$ such that

$$y_{n_k}\to y, \text{ as } n_k\to\infty,$$

$$|f_{n_k}^{-1}(y_{n_k})-f^{-1}(y)|>\epsilon_0\,,\quad \forall n_k\,.$$

Thus, for infinitely many n_k's, either $f_{n_k}^{-1}(y_{n_k}) > f^{-1}(y)+\epsilon_0$ or $f_{n_k}^{-1}(y_{n_k}) < f^{-1}(y)-\epsilon_0$. Without lost of generality, we consider only the former case in which $y_{n_k} > f_{n_k}(f^{-1}(y)+\epsilon_0)$ for infinitely many n_k's. Upon passing the limit $n_k\to\infty$, we obtain $y\ge f(f^{-1}(y)+\epsilon_0) > f(f^{-1}(y))$, which is a contradiction. □

Let us see how Proposition 2.8 follows from these propositions.

Proof of Proposition 2.8. By Proposition 3.5 (with $g\equiv 1$) we see that $\int_0^t W_\pi(X_\pi(s))ds$ converges almost surely to $\int_0^t \dot W(X(s))ds$ uniformly in t on compacts of $\mathbb{R}_+$. It is also obvious from the definitions of $S_{W_\pi}(x)$, $T_{W_\pi,B}(t)$, $X_\pi(t)$, and application of Lemma 3.8 that $X_\pi(t)$ converges almost surely to $X(t)$ uniformly in t on compact intervals of $\mathbb{R}_+$. From Proposition 3.4, it follows that $\mathcal{B}_\pi(t)$ converges almost surely to the process $\mathcal{B}$ defined in (2.25) uniformly on compact intervals of $\mathbb{R}_+$. By passing through the limit $|\pi|\to 0$ in (3.35), we see that X satisfies (1.1). In addition, by Lemma 3.2, for *every* π, $\mathcal{B}_\pi$ is the Brownian motion independent of W, hence, it is trivial to see that the limiting process $\mathcal{B}$ is also a Brownian motion independent of W. This finishes the proof. □

4. Itô formula - Proof of Theorem 2.9

Proof of Theorem 2.9. Let π be any partition of $\mathbb{R}$. Let W_π be the linear interpolation of W defined by (3.30). Denote $F_\pi(x)=\int_0^x f(y,W_\pi(y))dy+F(0)$ and $X_\pi(t)=S_{W_\pi}^{-1}\circ B\circ T_{W_\pi,B}^{-1}(t)$. We apply the time-changed Itô formula (3.26) for $F_\pi(X_\pi(t))=F_\pi\circ S_{W_\pi}^{-1}\circ B\circ T_{W_\pi,B}^{-1}(t)$, recall that in order to apply Itô formula we

first fix a realization of W and we also use the fact that B is a Brownian motion with respect to $\mathcal{F}^{B,W}$.

$$\begin{aligned}F_\pi(X_\pi(t)) = F(0) + \int_0^t (F_\pi \circ S_{W_\pi}^{-1})' \circ B \circ T_{W_\pi,B}^{-1}(s)\, dB \circ T_{W_\pi,B}^{-1}(s) \\ + \frac{1}{2}\int_0^t (F_\pi \circ S_{W_\pi}^{-1})'' \circ B \circ T_{W_\pi,B}^{-1}(s)\, dT_{W_\pi,B}^{-1}(s)\,.\end{aligned}$$

It is now easy to see that

$$(F_\pi \circ S_{W_\pi}^{-1})' \circ B \circ T_{W_\pi,B}^{-1}(s) = f(X_\pi(s), W_\pi(X_\pi(s)))e^{-W_\pi(X_\pi(s))}\,,$$

$$\begin{aligned}(F_\pi \circ S_{W_\pi}^{-1})'' \circ B \circ T_{W_\pi,B}^{-1}(s) &= \partial_x f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))} \\ &\quad + \partial_u f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))}\dot{W}_\pi(X_\pi(s)) \\ &\quad - f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))}\dot{W}_\pi(X_\pi(s))\,,\end{aligned}$$

and

$$dT_{W_\pi,B}^{-1}(s) = e^{2W_\pi(X_\pi(s))}ds\,.$$

Upon combining the above four identities, we obtain

$$\begin{aligned}F_\pi(X_\pi(t)) = F(0) &+ \int_0^t f(X_\pi(s), W_\pi(X_\pi(s)))\, d\mathcal{B}_\pi(s) \\ &+ \frac{1}{2}\int_0^t \partial_x f(X_\pi(s), W_\pi(X_\pi(s)))ds \\ &- \frac{1}{2}\int_0^t f(X_\pi(s), W_\pi(X_\pi(s)))\dot{W}_\pi(X_\pi(s))ds \\ &+ \frac{1}{2}\int_0^t f'(X_\pi(s), W_\pi(X_\pi(s)))\dot{W}_\pi(X_\pi(s))ds\,,\end{aligned} \tag{4.1}$$

where $\mathcal{B}_\pi(t) = \int_0^t e^{-W_\pi(X_\pi(s))}dB \circ T_{W_\pi,B}^{-1}(s)$ is a Brownian motion, as seen from Lemma 3.2. For every $\delta > 0$, from Proposition 3.5, we can choose a partition $\pi = \pi(\delta)$ such that

$$\int_0^t f(X_{\pi(\delta)}(s), W_{\pi(\delta)}(X_{\pi(\delta)}(s)))\dot{W}(X_{\pi(\delta)}(s))ds$$

and

$$\int_0^t f'(X_{\pi(\delta)}(s), W_{\pi(\delta)}(X_{\pi(\delta)}(s)))\dot{W}(X_{\pi(\delta)}(s))ds$$

converge to

$$\int_{\mathbb{R}} f(x, W(x))e^{-W(x)}L_B(T_{W,B}^{-1}(t), S_W(x))W(d^o x)$$

and

$$\int_{\mathbb{R}} f'(x, W(x))e^{-W(x)}L_B(T_{W,B}^{-1}(t), S_W(x))W(d^o x)$$

respectively as $\delta \downarrow 0$. In addition, since X_π and W_π converge to X and W, respectively, uniformly over compact intervals, with probability one, the integral $\int_0^t \partial_x f(X_\pi(s), W_\pi(X_\pi(s)))ds$ converges to $\int_0^t \partial_x f(X(s), W(X(s)))ds$. Hence, by

passing through the limit $\delta \downarrow 0$ in (4.1), it remains to show that the stochastic integral

$$\int_0^t f(X_\pi(s), W_\pi(X_\pi(s)))\, d\mathcal{B}_\pi(s)$$

converges to $\int_0^t f(X(s), W(X(s)))d\mathcal{B}(s)$ in probability as the mesh size of π shrinks to 0. For this purpose, we fix a continuous sample path of W and further denote $\tilde f(x) = f(x, W(x))$ and $\tilde f_\pi(x) = f(x, W_\pi(x))$. Since for fixed $t > 0$, X_π converges uniformly to X on $[0, t]$, for each $M > 0$ we can find a stopping time T_M such that

$$\sup_{s\le t}\sup_\pi |X_\pi(s \wedge T_M)| \le M\,.$$

Since X has finite range, we can also require $\lim_{M\to\infty} T_M = \infty$. Thus, it suffices to show the following limit in L^2

$$\lim_{|\pi|\downarrow 0}\int_0^{t\wedge T_M} \tilde f_\pi(X_\pi(s))d\mathcal{B}_\pi(s) = \int_0^{t\wedge T_M} \tilde f(X(s))d\mathcal{B}(s)\,.$$

Similarly to the proof of Proposition 3.4, it is equivalent to show

$$\lim_{|\pi|\to 0} \mathbb{E}^B\left[\int_0^{t\wedge T_M} \tilde f_\pi(X_\pi(s))\, d\mathcal{B}_\pi(s) \int_0^{t\wedge T_M} \tilde f(X(s))\, d\mathcal{B}(s)\right] = \mathbb{E}^B \int_0^{t\wedge T_M} |\tilde f(X(s))|^2\, ds\,. \tag{4.2}$$

Indeed, by the Itô isometry, the expectation on the left side equals to

$$\mathbb{E}^B\left[\int_0^{T^{-1}_{W_\pi,B}(t\wedge T_M)\wedge T^{-1}_{W,B}(t\wedge T_M)} (\tilde f_\pi \circ S^{-1}_{W_\pi})' \circ B(u) \cdot (\tilde f \circ S^{-1}_W)' \circ B(u)\, du\right].$$

It follows from Lemma 3.8 that with probability one

$$\begin{aligned}\lim_{|\pi|\to 0}&\int_0^{T^{-1}_{W_\pi,B}(t\wedge T_M)\wedge T^{-1}_{W,B}(t\wedge T_M)} (\tilde f_\pi \circ S^{-1}_{W_\pi})' \circ B(u) \cdot (\tilde f \circ S^{-1}_W)' \circ B(u) du\\ &= \int_0^{T^{-1}_{W,B}(t\wedge T_M)} |(\tilde f \circ S^{-1}_W)' \circ B(u)|^2\, du = \int_0^{t\wedge T_M} |\tilde f(X(s))|^2 ds\,.\end{aligned}$$

As in the proof of Proposition 3.4 we can use the Cauchy-Schwarz inequality and some changes of variables to see that

$$\begin{aligned}&\left(\int_0^{T^{-1}_{W_\pi,B}(t\wedge T_M)\wedge T^{-1}_{W,B}(t\wedge T_M)} (\tilde f_\pi \circ S^{-1}_{W_\pi})' \circ B(u) \cdot (\tilde f \circ S^{-1}_W)' \circ B(u)\, du\right)^2\\ &\le \int_0^{T^{-1}_{W_\pi,B}(t\wedge T_M)} |(\tilde f_\pi \circ S^{-1}_{W_\pi})' \circ B(u)|^2 du \int_0^{T^{-1}_{W,B}(t\wedge T_M)} |(\tilde f \circ S^{-1}_W)' \circ B(u)|^2 du\\ &= \int_0^{t\wedge T_M} |\tilde f_\pi(X_\pi(s))|^2 ds \int_0^{t\wedge T_M} |\tilde f(X(s))|^2 ds\\ &\le t^2 \sup_{|x|\le M} |\tilde f(x)|^4\,.\end{aligned}$$

We may use uniform integrability to get (4.2) and then to conclude the proof. □

5. Strong solution - Proof of Theorem 2.7

5.1. **Existence part of Theorem 2.7.** Because the methods proving existence and uniqueness are quiet different, we consider them separately. In this subsection, we focus on showing existence of a strong solution to equation (1.1). Throughout the current section, W is a (given) two-sided Brownian motioin and $\mathcal{B}$ is a (given) Brownian motion independent of W. We first seek for a Brownian motion B such that relation (2.25) is verified. For this purpose, we first prove the following result.

Lemma 5.1. *Let $\mathcal{B}$ be a Brownian motion and let W be two-sided Brownian motion independent of $\mathcal{B}$. Then, for P-a.s. W, the equation*

$$M(t) = \int_0^t e^{W\circ S_W^{-1}\circ M(u)} d\mathcal{B}(u)\,, \quad t\geq 0 \tag{5.3}$$

has unique strong solution $(M(t), t\geq 0)$ which has continuous sample paths.

Proof. First, we show the existence of the weak solution to (5.3). In fact, let $\tilde{B}$ be a Brownian independent from W. We define

$$\tilde{\mathcal{B}}(t) = \int_0^t e^{-W\circ S_W^{-1}\circ \tilde{B}\circ T^{-1}_{W,\tilde{B}}(s)} d\tilde{B}\circ T^{-1}_{W,\tilde{B}}(s)\,.$$

Then, it follows from Proposition 2.8 that $\tilde{\mathcal{B}}(t)$ is a Brownian motion, independent of W. Denote $\tilde{M} = \tilde{B}\circ T^{-1}_{W,\tilde{B}}$. Then $d\tilde{\mathcal{B}}(t) = e^{-W\circ S_W^{-1}\circ M(t)}dM(t)$ or $dM(t) = e^{W\circ S_W^{-1}\circ M(t)}d\tilde{\mathcal{B}}(t)$. This means that $(\tilde{M}, \tilde{B})$ is a weak solution to equation (5.3).

Let us prove the pathwise uniqueness for equation(5.3). Note that by the classical Lévy theorem W satisfies the following modulus of continuity condition: for each $n\geq 1$,

$$\begin{aligned} |W(x,\omega)-W(x',\omega)| \quad &\leq \quad c_n(\omega)\log(|x-x'|)\sqrt{|x-x'|} \qquad \forall x,x'\in[-n,n]\ , \\ &\qquad\qquad \text{for some } c_n(\omega)\geq 0,\ \text{for } P-\text{a.s. }\omega. \end{aligned}\tag{5.4}$$

Thus we can find a set $A\subset\Omega$ with $P(A)=1$, such that, for all $\omega\in A$, the following holds: for any $n\geq 1$, there exists $c_n(\omega)\geq 0$, such that

$$|W(x,\omega)-W(x',\omega)|\leq \rho_n(x,x'),\ \ \forall x,x'\in[-n,n]\,,$$

where $\rho_n(x,x') := c_n(\omega)\log(|x-x'|)\sqrt{|x-x'|}$. Fix arbitrary $\omega\in A$. For any $k\geq 1$, we define

$$\phi_k(z) = \phi_k(z,\omega) = e^{W(S_W^{-1}(-k\vee(z\wedge k)),\omega)} \tag{5.5}$$

and consider the following stochastic differential equation

$$M_k(t) \quad = \quad \int_0^t \phi_k(M_k(u))d\mathcal{B}(u)\,. \tag{5.6}$$

Note that

$$\begin{aligned} \int_{0+}^1 (\sqrt{|\log(u)u|})^{-2})\,du &= -\int_{0+}^1 (\log(u))^{-1}\,d(\log u) \\ &= \int_1^\infty \frac{1}{v}\,dv = \infty. \end{aligned}\tag{5.7}$$

We now take

$$n(k,w) = \lfloor |S_W^{-1}(k)| + |S_W^{-1}(-k)| + 1\rfloor\,,$$

where $\lfloor a \rfloor$ denotes the integer part of a. Then

$$|W(x,\omega)-W(x',\omega)|\le c_n(\omega)\log(|x-x'|)\sqrt{|x-x'|}$$

for all x, x' in the interval $[-n(k,w), n(k,w)]$. Define

$$S^*(\omega)=\sup_{|x|\le|S_W^{-1}(-k)|+|S_W^{-1}(k)|}(e^{W(x,\omega)}+e^{-W(x,\omega)}).$$

Then

$$\begin{aligned}|\phi_k(z)-\phi_k(z')|&\le S^*|W(S_W^{-1}(-k\vee(z\wedge k)))-W(S_W^{-1}(-k\vee(z'\wedge k)))|\\&\le S^*\rho(|S_W^{-1}(-k\vee(z\wedge k))-S_W^{-1}(-k\vee(z'\wedge k))|).\end{aligned}$$

Note that S_W^{-1} is Lipschitz function and we can easily derive:

$$|S_W^{-1}(-k\vee(z\wedge k))-S_W^{-1}(-k\vee(z'\wedge k))|\le S^*|z-z'|,$$

and hence

$$|\phi_k(z)-\phi_k(z')|\le S^*\rho(S^*|z-z'|).$$

This together with (5.7) implies the pathwise uniqueness of the equation (5.6) by standard Yamada-Watanabe criterion (see [11], Chapter IV, Theorem 3.2).

Now, let M^1 and M^2 be two continuous solutions to (5.3). Define the following stopping times:

$$\begin{aligned}T_k^{M_1,W}&=\inf\{t\ge0:\ M^1(t)=S_W(k)\text{ or }M^1(t)=S_W(-k)\},\\T_k^{M_2,W}&=\inf\{t\ge0:\ M^2(t)=S_W(k)\text{ or }M^2(t)=S_W(-k)\},\\\widetilde{T}_k^W&=\min\left(T_k^{M_1,W},T_k^{M_2,W}\right).\end{aligned}$$

Since the processes $(M^1(t), t\ge 0)$ and $(M^2(t), t\ge 0)$ have continuous sample paths, we see $\widetilde{T}_k^W\uparrow\infty$ a.s. when $k\to\infty$. When $t\le\widetilde{T}_k^W$, both $(M^1(t), t\ge0)$ and $(M^2(t), t\ge 0)$ satisfy (5.6). Thus $M^1(t)=M^2(t)$ when $t\le\widetilde{T}_k^W$. Passing through the limit $k\to\infty$ yields the strong uniqueness of the equation (5.3).

Finally, because weak existence and strong uniqueness together imply strong existence, we see that the equation (5.3) has a unique strong solution. □

We are now ready to prove the existence part of Theorem 2.7.

Proof of existence part of Theorem 2.7. Let M be the unique strong solution to equation (5.3). Define a stopping $\tau(t)$ so that

$$\int_0^{\tau(t)}e^{2W\circ S_W^{-1}\circ M(s)}ds=t\,. \tag{5.8}$$

We note that if M and W are provided, τ is uniquely determined by (5.8) because the map $u\mapsto\int_0^u e^{2W\circ S_W^{-1}\circ M(s)}ds$ is strictly increasing on $\mathbb{R}_+$. We define $B=M\circ\tau$. It follows from (5.3) that

$$\langle B\rangle_t=\int_0^{\tau(t)}e^{2W\circ S_W^{-1}\circ M(s)}ds=t\,.$$

Thus, from Lévy's characterization theorem, B is a Brownian motion. In addition, the relation (5.8) is equivalent to

$$\tau(t)=\int_0^t e^{-2W\circ S_W^{-1}\circ M\circ\tau(s)}ds\,.$$

Hence, taking into account the relation $M \circ \tau = B$, we have

$$\tau(t) = \int_0^t e^{-2W \circ S_W^{-1} \circ B(s)} ds = T_{W,B}(t). \tag{5.9}$$

From here and the equation (5.3) it follows that B and $\mathcal{B}$ satisfy the relation (2.25). In addition, similar to the proof of Proposition 2.8 it is clear that B is independent of W.

We now define $X = S_W^{-1} \circ B \circ T_{W,B}^{-1}$. Then Proposition 2.8 shows that X is a weak solution to (1.1) since we have shown that $\mathcal{B}$ and B satisfy the relation (2.25). Now by (5.9) we get that

$$X = S_W^{-1} \circ B \circ \tau^{-1} = S_W^{-1} \circ M, \tag{5.10}$$

where the last equality follows by the definition of B. Since M is the unique strong solution to (5.3), we get that M is adapted to filtration $\mathcal{F}^{\mathcal{B},W}$, and hence by (5.10) X is also adapted to filtration $\mathcal{F}^{\mathcal{B},W}$. This finishes the proof that X is a strong solution to the equation (1.1). ☐

5.2. **Uniqueness part of Theorem 2.7.** To show uniqueness for strong solutions of (1.1), we rely on Itô formula, Theorem 2.9.

Proof of uniqueness part of Theorem 2.7. Let $\mathcal{B}$ be a Brownian motion independent of W. We would like to show that X constructed in the proof of the existence part of Theorem 2.7 is indeed the unique strong solution to the equation (1.1). Let $\widetilde{X}$ be another strong solution, and let $\widetilde{B}$ the corresponding Brownian motion in the Itô-McKean representation, that is

$$\widetilde{X} = S_W^{-1} \circ \widetilde{B} \circ T_{W,\widetilde{B}}^{-1}. \tag{5.11}$$

Here, as usual,

$$T_{W,\widetilde{B}}(t) = \int_0^t e^{-2W \circ S_W^{-1}(\widetilde{B}(s))} ds\,, \tag{5.12}$$

or alternatively $T_{W,\widetilde{B}}(t)$ satisfies

$$\int_0^{T_{W,\widetilde{B}}(t)} e^{2W \circ \widetilde{X}(s)} ds = t\,. \tag{5.13}$$

The advantage of the later definition is that $T_{W,\widetilde{B}}(t)$ is given only via $\widetilde{X}$. By a simple transformation one can see that $\widetilde{B}$ can be expressed via $\widetilde{X}$ as

$$\widetilde{B}(t) = S_W \circ \widetilde{X} \circ T_{W,\widetilde{B}}. \tag{5.14}$$

Now we would like to express $\widetilde{B}$ as a solution to certain stochastic equation driven by $\mathcal{B}$. To this end we apply Itô formula from Theorem 2.9 to the function $S_W(x) = \int_0^x e^{W(y)} dy$. However, we cannot do it directly, since $x \mapsto e^x$ does not have bounded derivatives. Therefore an approximation is needed. Let R be a fixed positive number. Let f_R be a C^3-function with bounded derivatives such that $f_R(x) = e^x$ for every $x \in [-R, R]$ and $f_R = 0$ outside $[-R-1, R+1]$. We then apply Itô

formula from Theorem 2.9 to the function $F_R(x) = \int_0^x f_R(W(y))dy$ to get

$$F_R(\widetilde{X}(t)) = \int_0^t f_R(W(\widetilde{X}(s)))d\mathcal{B}(s) - \frac{1}{2}\int_{-\infty}^{\infty} f_R(W(x))L_{\widetilde{X}}(t,x)W(d^o x) + \frac{1}{2}\int_{-\infty}^{\infty} f_R'(W(x))L_{\widetilde{X}}(t,x)W(d^o x)\,.$$

Since $\widetilde{X}$ has continuous sample paths, $L_{\widetilde{X}}(t,\cdot)$ vanishes outside of a compact interval (independent from R). We can pass easily to the limit, as $R\to\infty$, and obtain

$$S_W(\widetilde{X}(t)) = \int_0^t e^{W(\widetilde{X}(s))}d\mathcal{B}(s). \tag{5.15}$$

The previous equation, (5.14) and (5.11) imply

$$\begin{aligned}\widetilde{B}(t) &= \int_0^{T_{W,\widetilde{B}}(t)} e^{W(\widetilde{X}(s))}d\mathcal{B}(s)\\ &= \int_0^{T_{W,\widetilde{B}}(t)} e^{W\circ S_W^{-1}\circ\widetilde{B}\circ T_{W,\widetilde{B}}^{-1}(s)}d\mathcal{B}(s)\end{aligned} \tag{5.16}$$

Then we immediately obtain

$$\widetilde{B}(T_{W,\widetilde{B}}^{-1}(t)) = \int_0^t e^{W\circ S_W^{-1}\circ\widetilde{B}\circ T_{W,\widetilde{B}}^{-1}(s))}d\mathcal{B}(s).$$

Thus $(\widetilde{B}\circ T_{W,\widetilde{B}}^{-1}(t), t\geq 0)$ satisfies (5.3). However, Lemma 5.1 states that the equation (5.3) has the unique strong solution. That is, if $M(t) = \widetilde{B}\circ T_{W,\widetilde{B}}^{-1}(t)$ then M is uniquely determined from the equation (5.3). In addition, upon comparing (5.13) with (5.8), we see that $T_{W,\widetilde{B}}(t) = \tau(t)$ where $\tau(t)$ is uniquely defined by (5.8). Note that both M and τ are solutions of equations ((5.3) and (5.8) respectively) which do not depend on *particular* solution $\widetilde{X}$ for (1.1). Then we have

$$\begin{aligned}\widetilde{B} &= M\circ\tau\\ &= B,\quad \text{a.s.}\end{aligned}$$

where B is the Brownian motion constructed in the proof of the existence part of Theorem 2.7. This and (5.11) imply that

$$\widetilde{X} = X,\quad \text{a.s.}$$

and uniqueness follows. □

6. Proof of Proposition 2.1

We have the following decomposition

$$\int_a^b g(x,W(x))L_B(\xi,S_W(x))\dot{W}_\pi(x)dx = I_1+I_2+I_3+I_4$$

where

$$I_1=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}[g(x,W(x))-g(x_k,W(x))]L_B(\xi,S_W(x))\frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k}dx\,,$$

$$I_2=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}g(x_k,W(x))[L_B(\xi,S_W(x))-L_B(\xi,S_W(x_k))]\frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k}dx\,,$$

$$I_3=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}[g(x_k,W(x))-g(x_k,W(x_k))]L_B(\xi,S_W(x_k))\frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k}dx\,,$$

$$I_4=\sum_{k=0}^{n-1}g(x_k,W(x_k))L_B(\xi,S_W(x_k))[W(x_{k+1})-W(x_k)]\,.$$

From the Cauchy-Schwarz inequality we see that I_1^2 is at most

$$(b-a)\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|g(x,W(x))-g(x_k,W(x))|^2|L_B(\xi,S_W(x))|^2\frac{[W(x_{k+1})-W(x_k)]^2}{(x_{k+1}-x_k)^2}dx\,.$$

Taking expectation and applying the Hölder inequality and (2.13) we obtain

$$\mathbb{E}I_1^2\lesssim\sum_{k=0}^{n-1}|x_{k+1}-x_k|^{2\lambda}\lesssim|\pi|^{2\lambda-1}$$

which implies $\mathbb{E}I_1^2$ goes to 0 since $\lambda>1/2$.

Denote each term in the expression of I_2 by I_{2k}. Then

$$\mathbb{E}(I_2^2)=\sum_{k=0}^{n-1}\mathbb{E}(I_{2k}^2)+\sum_{k\neq j}\mathbb{E}(I_{2k}I_{2j})=:I_{2,1}+I_{2,2}\,.$$

From the Cauchy-Schwarz inequality we see that $I_{2,1}$ is at most

$$\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\mathbb{E}\left\{|g(x_k,W(x))|^2\left[L_B(\xi,S_W(x))-L_B(\xi,S_W(x_k))\right]^2\frac{[W(x_{k+1})-W(x_k)]^2}{x_{k+1}-x_k}\right\}dx$$

By conditioning on the σ-algebra generated by W (namely taking the expectation with respect to the Brownian motion B first) and applying (3.39) with $\beta=1/2$, we see that

$$I_{2,1}\lesssim\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\mathbb{E}|g(x_k,W(x))|^2\left[|S_W(x)-S_W(x_k)|\,\frac{(W(x_{k+1})-W(x_k))^2}{x_{k+1}-x_k}\right]dx.$$

which is majorized by a constant multiple of $|\pi|$. It follows that $\lim_{|\pi|\to0}I_{2,1}=0$. If $k\neq j$ and if $x\in[x_j,x_{j+1})$ and $z\in[x_k,x_{k+1})$, then the intervals $[S_W(x_j),S_W(x))$ and $[S_W(x_k),S_W(z))$ are disjoint. Then we have from (3.41) with $\alpha=0$,

$$\begin{aligned}&\mathbb{E}[L_B(\xi,S_W(x))-L_B(\xi,S_W(x_j))][L_B(\xi,S_W(z))-L_B(\xi,S_W(x_k))]\\&\le\mathbb{E}|S_W(x)-S_W(x_j)||S_W(z)-S_W(x_k)|\,.\end{aligned}$$

Therefore, together with (2.13), we have

$$I_{2,2} \lesssim \sum_{j<k} \int_{x_j}^{x_{j+1}} \int_{x_k}^{x_{k+1}} \mathbb{E}\Bigg[e^{\theta|W(x)|+\theta|W(z)|} |S_W(x)-S_W(x_k)|\, |S_W(z)-S_W(x_k)| \left| \frac{W(x_{j+1})-W(x_j)}{x_{j+1}-x_j} \frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k} \right| \Bigg] dxdz\,.$$

It is now easy to check that $I_{2,2}$ converges to 0, hence so does I_2.

Using the Taylor expansion, we have

$$g(x_k, W(x)) - g(x_k, W(x_k)) = \partial_u g(x_k, W(x))(W(x) - W(x_k)) + R_k(x)$$

with $\sup_{0\le x\le y} \mathbb{E}|R_k(x)|^p \le C_p |x_{k+1}-x_k|^p$. Hence, we can decompose $I_3 = I_{3,1} + I_{3,3} + I_{3,3}$, where

$$I_{3,1} = \sum_{k=0}^{n-1} \left[\int_{x_k}^{x_{k+1}} (W(x)-W(x_k))dx \frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k} - \frac{1}{2}(x_{k+1}-x_k) \right] \times \partial_u g(x_k, W(x_k)) L_B(\xi, S_W(x_k))\,,$$

$$I_{3,2} = \frac{1}{2} \sum_{k=0}^{n-1} \partial_u g(x_k, W(x_k)) L_B(\xi, S_W(x_k))(x_{k+1}-x_k)\,,$$

and

$$I_{3,3} = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} R_k(x) dx\, L_B(\xi, S_W(x_k)) \frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k}\,.$$

$I_{3,1}$ is a sum of martingale difference. It is easy to see that

$$\begin{aligned}
\mathbb{E}(I_{3,1})^2 &\le \sum_{k=0}^{n-1} \mathbb{E}\left[|\partial_u g(x_k, W(x_k))|^2 L_B^2(\xi, S_W(x_k)) \right] \\
&\quad \times \left[\int_{x_k}^{x_{k+1}} (W(x)-W(x_k))dx \frac{W(x_{k+1})-W(x_k)}{x_{k+1}-x_k} - \frac{1}{2}(x_{k+1}-x_k) \right]^2 \\
&\le C \sum_{k=0}^{n-1} (x_{k+1}-x_k)^2 \to 0\,.
\end{aligned}$$

$I_{3,2}$ is the Riemann sum of the integral $\frac{1}{2}\int_a^b \partial_u g(x, W(x)) L_B(\xi, S_W(x))dx$. A straightforward estimation yields that $I_{3,3}$ converges to 0 in L^2. Hence, we have I_3 converges to $\frac{1}{2}\int_a^b \partial_u g(x, W(x)) L_B(\xi, S_W(x))dx$ in L^2. By standard Itô calculus, we see that I_4 converges in L^2 to the Itô integral $\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(dx)$. □

7. Proof of Proposition 3.4

From Doob's maximal inequality, it suffices to show

$$\lim_{|\pi|\to 0} \mathbb{E}|\mathcal{B}_\pi(t) - \mathcal{B}(t)|^2 = 0\,,$$

for every fixed $t > 0$. We write

$$\begin{aligned}\mathcal{B}_\pi(t) &= \int_0^t e^{-W_\pi(X_\pi(s))} dB \circ T^{-1}_{W_\pi,B}(s) \\ &= \int_0^{T^{-1}_{W_\pi,B}(t)} e^{-W_\pi(X_\pi \circ T^{-1}_{W_\pi,B}(u))} dB(u) \\ &= \int_0^{T^{-1}_{W_\pi,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u))} dB(u)\end{aligned}$$

and similarly

$$\mathcal{B}(t) = \int_0^{T^{-1}_{W,B}(t)} e^{-W(S^{-1}_{W,B} \circ B(u))} dB(u)\,.$$

By a change of variable (similar to the one used in the proof of Lemma 3.2), we can immediately get that the quadratic variation of $\mathcal{B}$ is given by

$$\int_0^{T^{-1}_{W,B}(t)} e^{-2W(S^{-1}_{W,B} \circ B(u))} du = \int_0^t e^{-2W(S^{-1}_{W,B} \circ B \circ T^{-1}_{W,B}(s))} dT^{-1}_{W,B}(s) = t\,, \tag{7.1}$$

and hence $\mathcal{B}$ is a Brownian motion with respect to $\{\mathcal{F}^{B,W}_{T^{-1}_{W,B}(t)}\}_{t\geq 0}$. In addition, Lemma 3.2 asserts that $\mathcal{B}_\pi$ is a Brownian motion with respect to $\{\mathcal{F}^{B,W}_{T^{-1}_{W,B}(t)}\}_{t\geq 0}$. Since $\mathcal{B}_\pi$ and $\mathcal{B}$ are square integrable martingales we get

$$\begin{aligned}\mathbb{E}\left(\mathcal{B}_\pi(t)\mathcal{B}(t)\right) &= \mathbb{E}\left[\int_0^{T^{-1}_{W_\pi,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u))} dB(u) \int_0^{T^{-1}_{W,B}(t)} e^{-W(S^{-1}_{W,B} \circ B(u))} dB(u)\right] \\ &= \mathbb{E}\left[\int_0^{T^{-1}_{W_\pi,B}(t) \wedge T^{-1}_{W,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u)) - W(S^{-1}_{W,B} \circ B(u))} du\right]. \qquad (7.2)\end{aligned}$$

From Lemma 3.8, $T^{-1}_{W_\pi,B}$ and $S^{-1}_{W_\pi}$ converge uniformly over finite intervals, almost surely, to $T^{-1}_{W,B}$ and S^{-1}_{W} respectively. Hence, for each $t \geq 0$,

$$\begin{aligned}&\int_0^{T^{-1}_{W_\pi,B}(t) \wedge T^{-1}_{W,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u)) - W(S^{-1}_{W,B} \circ B(u))} du \qquad (7.3) \\ &\qquad\qquad \to \int_0^{T^{-1}_{W,B}(t)} e^{-2W(S^{-1}_{W,B} \circ B(u))} du = t,\end{aligned}$$

with probability one, as $|\pi| \to 0$. The last equality follows from (7.1).

Now, by first applying Cauchy-Schwarz inequality, and then equalities (3.38) and (7.1) we get

$$\begin{aligned}&\left(\int_0^{T^{-1}_{W_\pi,B}(t) \wedge T^{-1}_{W,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u)) - W(S^{-1}_{W,B} \circ B(u))} du\right)^2 \\ &\qquad \leq \int_0^{T^{-1}_{W_\pi,B}(t)} e^{-2W_\pi(S^{-1}_{W_\pi,B} \circ B(u))} du \int_0^{T^{-1}_{W,B}(t)} e^{-2W(S^{-1}_{W,B} \circ B(u))} du = t^2\,.\end{aligned}$$

The above bound implies uniform integrability of random variables

$$\int_0^{T^{-1}_{W_\pi,B}(t) \wedge T^{-1}_{W,B}(t)} e^{-W_\pi(S^{-1}_{W_\pi,B} \circ B(u)) - W(S^{-1}_{W,B} \circ B(u))} du,$$

and hence by (7.3) we get that the right hand side of (7.2) converges to t, and this immediately implies that $\lim_{|\pi|\to 0} \mathbb{E}\mathcal{B}_\pi(t)\mathcal{B}(t) = t$. Therefore,

$$\begin{aligned}\mathbb{E}(\mathcal{B}_\pi(t) - \mathcal{B}(t))^2 &= \mathbb{E}(\mathcal{B}_\pi(t)^2) + \mathbb{E}(\mathcal{B}(t)^2) - 2\mathbb{E}(\mathcal{B}_\pi(t)\mathcal{B}(t)) \\ &= 2t - 2\mathbb{E}(\mathcal{B}_\pi(t)\mathcal{B}(t))\end{aligned}$$

converges to 0 as $|\pi| \to 0$. $\square$

8. Proof of Proposition 3.6

Let $p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ be the heat kernel and $\mathfrak{S}_m$ denote the symmetric group of permutations of $\{1, 2, \cdots, m\}$. It is easy to verify that for generic points $u_1, \ldots, u_m$ in $\mathbb{R}$, we have

$$\mathbb{E}\prod_{j=1}^m L_B([\xi,\eta], u_j) = \sum_{\sigma\in\mathfrak{S}_m} \int_{D_m} \prod_{j=1}^m p(s_j - s_{j-1}, u_{\sigma_j} - u_{\sigma_{j-1}})\, d\bar{s}\,, \tag{8.1}$$

where D_m is the domain $\{\bar{s} \in [\xi,\eta]^m : \xi < s_1 < \cdots < s_m < \eta\}$, $d\bar{s} = ds_1 \cdots ds_m$, and $u_0 = 0$ by convention. (8.1) is in fact the so-called Kac moment formula (see Marcus-Rosen's book [14]).

To use (8.1) to compute the two moments in (3.39) and (3.41), we need to introduce some notations. As introduced in [10], for $k = 1, \ldots, n$ and $x \in \mathbb{R}$, $V_k(x)$ denotes the substitution operator, i.e. for a generic function $f = f(u_1, \cdots, u_n)$, $V_k(x)f(u) = f(u_1, \cdots, u_{k-1}, x, u_{k+1}, \cdots, u_m)$. It is clear that if $f(u)$ is a random process, then

$$\mathbb{E}V_k(x)f(u) = \mathbb{E}f(u_1, \cdots, u_{k-1}, x, u_{k+1}, \cdots, u_m) = V_k(x)\mathbb{E}f(u)\,.$$

Thus the operator V_k commutes with the expectation operator.

For any points $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ in $\mathbb{R}$, we denote $\bar{x} = (x_1, \cdots, x_m)$ and $\bar{y} = (y_1, \cdots, y_m)$. The notation $[\bar{x}, \bar{y}]$ denotes the rectangle $[x_1, y_1] \times \cdots \times [x_m, y_m]$ in $\mathbb{R}^m$. The operator $\square^m([\bar{x},\bar{y}])$ is defined as $\square^m([\bar{x},\bar{y}]) := \prod_{k=1}^m [V_k(y_k) - V_k(x_k)]$. When applied to an m-multivariate function, $\square^m([\bar{x},\bar{y}])$ is the rectangular increment of the function over the rectangle $[\bar{x},\bar{y}]$. In particular, when $f(x) = f(x_1)\cdots f(x_m)$, then $\square^m([\bar{x},\bar{y}])f = \prod_{k=1}^m [f(y_k) - f(x_k)]$. Moreover, for sufficiently smooth function f, the rectangular increment of f can be computed as follows

$$\square^m([\bar{x},\bar{y}])f = \int_{[x,y]} \frac{\partial^m}{\partial z_1 \partial z_2 \cdots \partial z_m} f(\bar{z})\, d\bar{z}\,. \tag{8.2}$$

With these notations, we can write $\prod_{k=1}^m (L_B([\xi,\eta], y_k) - L_B([\xi,\eta], x_k))$ as follows

$$\prod_{k=1}^m (L_B([\xi,\eta], y_k) - L_B([\xi,\eta], x_k)) = \square^m([\bar{x},\bar{y}]) \prod_{j=1}^m L_B([\xi,\eta], u_j)\,.$$

Notice that the operator $\Box$ also commutes with the expectation operator. In particular, when combined with (8.1), we obtain the formula

$$\mathbb{E}\prod_{k=1}^{m}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right) = \sum_{\sigma\in\mathfrak{S}_m}\int_{D_m} d\bar{s}\,\Box^m([\bar{x},\bar{y}])\prod_{j=1}^{m}p(s_j-s_{j-1},u_{\sigma_j}-u_{\sigma_{j-1}})\,. \tag{8.3}$$

First, let us assume $x_1=\cdots=x_m=x$ and $y_1=\cdots=y_m=y$. Denote $\bar{x}_{\hat{m}}=(x_{\sigma_1},\cdots,x_{\sigma_{m-1}})$ and $\bar{x}_{\hat{m},\widehat{m-1}}=(x_{\sigma_1},\cdots,x_{\sigma_{m-2}})$ etc. From (8.3) it follows

$$\begin{aligned}
&\mathbb{E}\prod_{k=1}^{m}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\\
&=\sum_{\sigma\in\mathfrak{S}_m}\int_{D_m} d\bar{s}\,\Box^{m-1}([\bar{x}_{\hat{m}},\bar{y}_{\hat{m}}])\prod_{j=1}^{m-1}p(s_j-s_{j-1},u_{\sigma_j}-u_{\sigma_{j-1}})\\
&\qquad\left[p(s_m-s_{m-1},y-u_{\sigma_{m-1}})-p(s_m-s_{m-1},x-u_{\sigma_{m-1}})\right]\\
&=\sum_{\sigma\in\mathfrak{S}_m}\int_{D_m} d\bar{s}\,\Box^{m-2}([\bar{x}_{\hat{m},\widehat{m-1}},\bar{y}_{\hat{m},\widehat{m-1}}])\prod_{j=1}^{m-2}p(s_j-s_{j-1},u_{\sigma_j}-u_{\sigma_{j-1}})\\
&\qquad\Big\{\left[p(s_m-s_{m-1},y-y)-p(s_m-s_{m-1},x-y)\right]p(s_{m-1}-s_{m-2},y-u_{\sigma_{m-2}})\\
&\qquad\quad-\left[p(s_m-s_{m-1},y-x)-p(s_m-s_{m-1},x-x)\right]p(s_{m-1}-s_{m-2},x-u_{\sigma_{m-2}})\Big\}\\
&=\sum_{\sigma\in\mathfrak{S}_m}\int_{D_m} d\bar{s}\,\Box^{m-2}([\bar{x}_{\hat{m},\widehat{m-1}},\bar{y}_{\hat{m},\widehat{m-1}}])\prod_{j=1}^{m-2}p(s_j-s_{j-1},u_{\sigma_j}-u_{\sigma_{j-1}})\\
&\qquad\left[p(s_{m-1}-s_{m-2},y-u_{\sigma_{m-2}})+p(s_{m-1}-s_{m-2},x-u_{\sigma_{m-2}})\right]\\
&\qquad\times\left[p(s_m-s_{m-1},0)-p(s_m-s_{m-1},x-y)\right]\,.
\end{aligned}$$

If we continue to apply the operator V this way, we shall obtain

$$\begin{aligned}
&\mathbb{E}\left[L([\xi,\eta],x)-L([\xi,\eta],y)\right]^{2n}\\
&=(2n)!\int_{D_{2n}}\prod_{k=2}^{2n}\left[p(s_k-s_{k-1},0)+(-1)^{k+1}p(s_k-s_{k-1},x-y)\right]\left[p(s_1,x)+p(s_1,y)\right]d\bar{s}\,.
\end{aligned}\tag{8.4}$$

The estimate (3.39) follows from (8.4) and the following inequality

$$\int_a^b\int_s^b (t-s)^\gamma[p(t-s,0)-p(t-s,x-y)][p(s-a,0)+p(s-a,x-y)]dtds \le c_{\beta,\gamma}|b-a|^{\gamma+1-\beta}|x-y|^{2\beta}\,,$$

is valid for all $\beta\in[0,1/2]$ and $\gamma\ge 0$.

Now we assume a condition which is slightly more restricted than (3.40):

$$x_1<y_1<x_2<y_2<\cdots<x_{2n}<y_{2n}\,. \tag{8.5}$$

The functions $p(t,x)$ and all its partial derivatives are continuously differentiable on the any interval $(t,x)\in[0,\infty)\times(-\infty,-a]\cup[a,\infty)$ for any positive a. Thus

the function on the right hand side of (8.1) is continuously differentiable on the $[\bar{x},\bar{y}]$ satisfying (8.5). Using the equations (8.2), (8.3) and interchanging order of integrations, we have

$$\begin{aligned}(8.6)\quad &\mathbb{E}\prod_{k=1}^{m}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\\ &\qquad=\sum_{\sigma\in\mathfrak{S}_m}\int_{D_m}d\bar{s}\int_{[\bar{x},\bar{y}]}d\bar{z}\,\frac{\partial^m}{\partial z_1\cdots\partial z_m}\prod_{j=1}^{m}p(s_j-s_{j-1},z_{\sigma_j}-z_{\sigma_{j-1}}).\end{aligned}$$

Notice that each partial derivative $\partial/\partial z_{\sigma_j}$ contributes one derivative to either $p(s_j-s_{j-1},z_{\sigma_j}-z_{\sigma_{j-1}})$ or $p(s_{j+1}-s_j,z_{\sigma_{j+1}}-z_{\sigma_j})$. We record the results by a binary index e_j, $e_j=1$ represents the former case, $e_j=0$ represents the later case. Moreover, if the later case happens, it also contributes a factor -1. Since z_m only appears in the last term $p(s_m-s_{m-1},z_{\sigma_m}-z_{\sigma_{m-1}})$, we must have the restriction $e_m=1$. Thus, we can write (8.6) as

$$\begin{aligned}(8.7)\quad &\mathbb{E}\prod_{k=1}^{m}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\\ &=\sum_{\sigma\in\mathfrak{S}_m}\sum_{\bar{e}\in\mathfrak{E}}\operatorname{sgn}(\bar{e})\int_{D_m}d\bar{s}\int_{[\bar{x},\bar{y}]}d\bar{z}\prod_{j=1}^{m}p^{(e_j+1-e_{j-1})}(s_j-s_{j-1},z_{\sigma_j}-z_{\sigma_{j-1}})\,,\end{aligned}$$

where $\mathfrak{E}$ denotes all the m-tuple $\bar{e}=(e_1,\dots,e_m)\in\{0,1\}^m$ such that $e_m=1$ and $\operatorname{sgn}(\bar{e})$ is the sign of $\bar{e}$, defined by $\operatorname{sgn}\bar{e}:=(-1)^{\sum_{j=1}^m(1-e_j)}$ and $e_0=1$ by convention.

For instance, in the case $m=4$, when σ is the identity map in $\mathfrak{S}_4$, the integrand in (8.7) is

$$(8.8)\quad\begin{aligned}&-p''_{s_4-s_3}(z_4-z_3)p'_{s_3-s_2}(z_3-z_2)p'_{s_2-s_1}(z_2-z_1)p_{s_1}(z_1)\\ &+p'_{s_4-s_3}(z_4-z_3)p''_{s_3-s_2}(z_3-z_2)p'_{s_2-s_1}(z_2-z_1)p_{s_1}(z_1)\\ &+p''_{s_4-s_3}(z_4-z_3)p_{s_3-s_2}(z_3-z_2)p''_{s_2-s_1}(z_2-z_1)p_{s_1}(z_1)\\ &-p'_{s_4-s_3}(z_4-z_3)p'_{s_3-s_2}(z_3-z_2)p''_{s_2-s_1}(z_2-z_1)p_{s_1}(z_1)\\ &+p''_{s_4-s_3}(z_4-z_3)p'_{s_3-s_2}(z_3-z_2)p_{s_2-s_1}(z_2-z_1)p'_{s_1}(z_1)\\ &-p'_{s_4-s_3}(z_4-z_3)p''_{s_3-s_2}(z_3-z_2)p_{s_2-s_1}(z_2-z_1)p'_{s_1}(z_1)\\ &-p''_{s_4-s_3}(z_4-z_3)p_{s_3-s_2}(z_3-z_2)p'_{s_2-s_1}(z_2-z_1)p'_{s_1}(z_1)\\ &+p'_{s_4-s_3}(z_4-z_3)p'_{s_3-s_2}(z_3-z_2)p'_{s_2-s_1}(z_2-z_1)p'_{s_1}(z_1)\,.\end{aligned}$$

Combining the estimate in Lemma 8.1 (below) with (8.7), we see that there exists a constant c_n depending only on n such that

$$(8.9)\qquad\left|\mathbb{E}\prod_{k=1}^{2n}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\right|\le c_n\prod_{j=1}^{2n}|x_j-y_j|\,.$$

An application (3.39) with $\beta=0$ yields

$$(8.10)\qquad\left|\mathbb{E}\prod_{k=1}^{2n}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\right|\le c_n|\eta-\xi|^n\,.$$

Now given $\alpha \in [0,1]$, an interpolating between (8.9) and (8.10) yields

$$\left|\mathbb{E}\prod_{k=1}^{2n}\left(L_B([\xi,\eta],y_k)-L_B([\xi,\eta],x_k)\right)\right| \le c_{\alpha,n}|\eta-\xi|^{n\alpha}\prod_{j=1}^{2n}|x_j-y_j|^{1-\alpha}\,.$$

This is (3.41) under the condition (8.5). The estimate (3.41) under the general condition (3.40) follows by a limiting argument since both sides of (3.41) are continuous function of x_k, y_k's. This finishes the proof of Proposition 3.6 modulo the proof of the following lemma which was used in the above proof.

Lemma 8.1. *Let $\bar{e} = (e_1, \dots, e_m)$ $(m \ge 2)$ be an m-tuple in $\{0,1\}^m$ such that $e_m = 1$ and we take $e_0 = 1$ by convention. Let u_k $(k = 1, 2, \dots, m)$ be non-zero real numbers and let D_m be the domain $\{\bar{s} \in [\xi,\eta]^m : \xi < s_1 < \cdots < s_m < \eta\}$. Then the following estimate holds*

$$\left|\int_{D_m}\prod_{j=1}^{m}p^{(e_j+1-e_{j-1})}(s_j-s_{j-1},u_j)\,d\bar{s}\right| \le 1\,. \tag{8.11}$$

Proof. We denote $\mathcal{L}$ the Laplace transform with respect to the t variable and put

$$J=\int_{D_m}\prod_{j=1}^{m}p^{(e_j-e_{j-1}+1)}(s_j-s_{j-1},u_j)\,d\bar{s}\,. \tag{8.12}$$

Let $*$ denote the convolution operator, i.e. for two functions f and g, $f*g(t) = \int_0^t f(s)g(t-s)ds$. Then we can rewrite J into the form

$$J=\int_{\xi}^{\eta}p^{(e_1)}(s_1,z_{\sigma_1})f(\eta-s_1)\,ds_1\,,$$

where f is the function defined by

$$f(t)=\int_0^t[p^{(e_2-e_1+1)}(\cdot,u_2)*\cdots*p^{(e_m-e_{m-1}+1)}(\cdot,u_m)](s)ds\,.$$

It is well known (see for example, [8], Formula 3.471 (9) and Formula 8.469 (3)) that

$$\mathcal{L}[p(\cdot,x)](s)=\frac{1}{\sqrt{2s}}e^{-|x|\sqrt{2s}}\,. \tag{8.13}$$

By taking derivative under the integral sign (noticing that we assume $x \neq 0$), we obtain

$$\mathcal{L}[p'(\cdot,x)](s)=-\mathrm{sgn}\,(x)e^{-|x|\sqrt{2s}}\,.$$

We further notice that $p'' = 2\partial_t p$, thus

$$\mathcal{L}[p''(\cdot,x)](s)=\sqrt{2s}e^{-|x|\sqrt{2s}}\,.$$

Writing all three formulas in one, for $k = 0, 1, 2$, we have

$$\mathcal{L}[p^{(k)}(\cdot,x)](s)=(\sqrt{2s})^{k-1}[-\mathrm{sgn}\,(x)]^k e^{-|x|\sqrt{2s}}\,. \tag{8.14}$$

Since convolution becomes product under Laplace transform, the Laplace transform of f is

$$\begin{aligned}\mathcal{L}[f](s) &= s^{-1}\prod_{j=2}^{m}(\sqrt{2s})^{e_j-e_{j-1}}[-\mathrm{sgn}\,(u_j)]^{e_j-e_{j-1}+1}\exp\left\{-|u_j|\sqrt{2s}\right\}\\ &= \sqrt{2}(\sqrt{2s})^{-1-e_1}\exp\left\{-\sqrt{2s}\sum_{j=2}^{m}|u_j|\right\}\prod_{j=2}^{m}[-\mathrm{sgn}\,(u_j)]^{e_j-e_{j-1}+1},\end{aligned}$$

where the factor s^{-1} comes from the fact that the Laplace transform of $\int_0^t f(r)dr$ is $s^{-1}\mathcal{L}f(s)$. To simplify notations, we will denote $|u|=\sum_{j=2}^m|u_j|$. We consider now two cases. Case 1: $e_1=0$. Inverting the Laplace transform, using (8.13), we see that

$$f(t)=\sqrt{2}\prod_{j=2}^{m}[-\mathrm{sgn}\,(u_j)]^{e_j-e_{j-1}+1}p(t,|u|)\,. \tag{8.15}$$

Thus

$$|J|\le\sqrt{2}\int_\xi^\eta p(s_1,u_1)p(\eta-s_1,|u|)ds_1\le 1/\sqrt{2}\,.$$

Case 2: $e_1=1$. We notice that

$$\mathcal{L}\left[\mathrm{erfc}\left(\frac{|x|}{\sqrt{2t}}\right)\right](s)=\frac{1}{s}e^{-|x|\sqrt{2s}}\,,$$

where $\mathrm{erfc}(z)$ is the complementary error function $\mathrm{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-y^2}dy$. Inverting the Laplace transform as in the former case, we obtain

$$f(t)=\prod_{j=2}^{m}[-\mathrm{sgn}\,(u_j)]^{e_j-e_{j-1}+1}\mathrm{erfc}\left(\frac{|u|}{\sqrt{2t}}\right)\,. \tag{8.16}$$

Thus if we use the fact that $0\le\mathrm{erfc}(z)\le 1$, we have

$$\begin{aligned}|J| &\le \int_\xi^\eta|p'(s_1,u_1)|\mathrm{erfc}\left(\frac{|u|}{\sqrt{2(\eta-s_1)}}\right)ds_1\\ &\le\int_\xi^\eta\frac{|u_1|}{\sqrt{2\pi s_1}}e^{-\frac{|u_1|^2}{2s_1}}\frac{ds_1}{s_1}\,.\end{aligned}$$

By the change of variable $t=\frac{|u_1|}{\sqrt{2s_1}}$, we see that $|J|\le\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt\le 1$. □

9. Proof of Proposition 3.5

To outline the strategy proving Proposition 3.5, let us first observe that using the representation $X_\pi=S_{W_\pi}^{-1}\circ B\circ T_{W_\pi,B}^{-1}$ we can write

$$\begin{aligned}&\int_0^t g(X_\pi(s),W_\pi\circ(X_\pi(s))\dot{W}_\pi(X_\pi(s))ds\\ &\qquad=\int_{-\infty}^\infty g(x,W_\pi(x))e^{-W_\pi(x)}L_B(T_{W_\pi,B}^{-1}(t),S_{W_\pi}(x))\dot{W}_\pi(x)\,dx\,.\end{aligned}$$

We observe that from Lemma 3.8, with probability one, $T^{-1}_{W_\pi,B}(\cdot)$ converges to $T^{-1}_{W,B}(\cdot)$ uniformly over compacts of $\mathbb{R}_+$. In addition, the function e^{-u} can be combined with $g(x,u)$. Therefore, to prove Proposition 3.5, it suffices to show

- For every function g satisfying conditions (2.13) and (2.14), with probability one, the process $\xi \mapsto \int_{-\infty}^{\infty} g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x)\, dx$ converges to $\int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x)$ uniformly over compact sets.

The remaining of this section is devoted to verify the previous statement. In what follows, $\{\ell_\pi(g,\xi), \xi \geq 0\}$ denote the process

$$\ell_\pi(g,\xi) = \int_{\mathbb{R}} g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x)\, dx\,, \tag{9.1}$$

which is well-defined for all continuous sample paths of W. For every compact set K, we denote

$$c_3(K) = c_1(K) + c_2(K)$$

where c_1 and c_2 are the constant in (2.13) and (2.14).

In subsection 9.1, we will truncate the processes ℓ_π and show the corresponding truncated processes converges uniformly. In subsection 9.2, the claim is verified completely via a gluing argument.

Let us remark that for all the results in this section holds, we employ the two estimates (3.39) and (3.41) for the local time of Brownian motion B, L_B.

9.1. **Convergence over bounded interval.** We consider an interval $[a,b]$ with length $L = b - a$. Let $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ with mesh size

$$\Delta = \max_{k=0,\dots,n-1} |x_{k+1} - x_k|.$$

We denote

$$\ell_\pi^{[a,b]}(g,\xi) = \int_a^b g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x) dx\,, \tag{9.2}$$

where as usual W_π is the linear interpolation of W associated with π.

We first decompose $\ell_\pi^{[a,b]}(g,\xi)$ as follows

$$\begin{aligned}
\ell_\pi^{[a,b]}(g,\xi) = &\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} g(x, W_\pi(x)) [L_B(\xi, S_{W_\pi}(x)) - L_B(\xi, S_{W_\pi}(x_k))] \dot{W}_\pi(x) dx \\
&+ \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} [g(x, W_\pi(x)) - g(x_k, W_\pi(x))] \dot{W}_\pi(x) dx \\
&+ \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} g(x_k, W_\pi(x)) \dot{W}_\pi(x) dx
\end{aligned}$$

Let G be a function such that $\partial_u G(x,u) = g(x,u)$. The integral inside the last summand can be computed as follows

$$\begin{aligned}
\int_{x_k}^{x_{k+1}} g(x_k, W_\pi(x)) \dot{W}_\pi(x) dx &= G(x_k, W(x_{k+1})) - G(x_k, W(x_k)) \\
&= \int_{x_k}^{x_{k+1}} g(x_k, W(x)) W(dx) + \frac{1}{2} \int_{x_k}^{x_{k+1}} \partial_u g(x_k, W(x)) dx\,,
\end{aligned}$$

where the last line follows from the classical Itô formula. Therefore, we can further decompose $\ell_\pi^{[a,b]}(g,\xi)$ as

$$\ell_\pi^{[a,b]}(g,\xi) = I_1(\xi) + I_2(\xi) + I_3(\xi) + I_4(\xi)$$

where

$$I_1(\xi) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} g(x, W_\pi(x))[L_B(\xi, S_{W_\pi}(x)) - L_B(\xi, S_{W_\pi}(x_k))]\dot W_\pi(x)dx\,,$$

$$I_2(\xi) = \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} [g(x, W_\pi(x)) - g(x_k, W_\pi(x))]\dot W_\pi(x)dx\,,$$

$$I_3(\xi) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} L_B(\xi, S_{W_\pi}(x_k))g(x_k, W(x))W(dx)\,,$$

$$I_4(\xi) = \frac{1}{2}\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \partial_u g(x_k, W(x))L_B(\xi, S_{W_\pi}(x_k))dx\,.$$

To simplify notation, we omit dependence of I_i's on g. For a generic function f on $\mathbb{R}$, we will denote

$$f([\xi,\eta]) \equiv f(\eta) - f(\xi),\ \forall \eta, \xi \in \mathbb{R}.$$

Lemma 9.1. *Suppose g satisfies the conditions in Proposition 3.5. There exist positive constants ϵ, γ, κ which does not depend on (a,b) such that the following estimates holds: for all $\eta, \xi \in \mathbb{R}_+$,*

$$\mathbb{E}|I_1([\xi,\eta])|^6 \lesssim c_1^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^{1+\epsilon}\Delta^\gamma\,, \tag{9.3}$$

$$\mathbb{E}|I_2([\xi,\eta])|^6 \lesssim c_2^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^{1+\epsilon}\Delta^\gamma\,, \tag{9.4}$$

$$\begin{aligned}\mathbb{E}|I_3([\xi,\eta]) - \int_a^b g(x, W(x))L_B([\xi,\eta], S_W(x))W(dx)|^6 \\ \lesssim c_3^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^{1+\epsilon}\Delta^\gamma\,,\end{aligned} \tag{9.5}$$

$$\begin{aligned}\mathbb{E}|I_4([\xi,\eta]) - \frac{1}{2}\int_a^b \partial_u g(x, W(x))L_B([\xi,\eta], S_W(x))dx|^6 \\ \lesssim c_3^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^{1+\epsilon}\Delta^\gamma\,,\end{aligned} \tag{9.6}$$

where the implied constants depend only on $b-a$. As a consequence, for all $\eta, \xi \in \mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}|\ell_\pi^{[a,b]}(g,[\xi,\eta]) - \int_a^b g(x, W(x))L_B([\xi,\eta], S_W(x))W(d^o x)|^6 \\ \lesssim c_3^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^{1+\epsilon}\Delta^\gamma\,.\end{aligned} \tag{9.7}$$

Proof. To deal with I_1, we denote

$$a_k = \int_{x_k}^{x_{k+1}} g(x, W_\pi(x))\left[L_B([\xi,\eta], S_{W_\pi}(x)) - L_B([\xi,\eta], S_{W_\pi}(x_k))\right]\dot W_\pi(x)dx\,.$$

Then

$$
\begin{aligned}
&\mathbb{E}I_1([\xi,\eta])^6 \\
&= \sum_{k_1=0}^{n-1} \mathbb{E}a_{k_1}^6 + 6\sum_{k_1\neq k_2} \mathbb{E}a_{k_1}^5 a_{k_2} + 15\sum_{k_1,k_2} \mathbb{E}a_{k_1}^4 a_{k_2}^2 + 30\sum_{k_1,k_2,k_3} \mathbb{E}a_{k_1}^4 a_{k_2} a_{k_3} \\
&\quad + 20\sum_{k_1,k_2} \mathbb{E}a_{k_1}^3 a_{k_2}^3 + 60\sum_{k_1,k_2,k_3} \mathbb{E}a_{k_1}^3 a_{k_2}^2 a_{k_3} + 120\sum_{k_1,k_2,k_3,k_4} \mathbb{E}a_{k_1}^3 a_{k_2} a_{k_3} a_{k_4} \\
&\quad + 90\sum_{k_1,k_2,k_3} \mathbb{E}a_{k_1}^2 a_{k_2}^2 a_{k_3}^2 + 180\sum_{k_1,k_2,k_3,k_4} \mathbb{E}a_{k_1}^2 a_{k_2}^2 a_{k_3} a_{k_4} \\
&\quad + 360\sum_{k_1,k_2,k_3,k_4,k_5} \mathbb{E}a_{k_1}^2 a_{k_2} a_{k_3} a_{k_4} a_{k_5} + 6!\sum_{k_1,k_2,k_3,k_4,k_5,k_6} \mathbb{E}a_{k_1} a_{k_2} a_{k_3} a_{k_4} a_{k_5} a_{k_6}
\end{aligned}
$$

where the indices $k_1,\dots,k_6$ are pairwise disjoint if they appear under the same summation notation. Among these sums, the most difficult term to estimate is the last one. All other sums can be handled by mean of the Hölder inequality and (3.39) (similar to the method of estimating A below). To illustrate our method while maintain a decent length of the paper, we will give detailed estimates for the two sums

$$
\begin{aligned}
A &= \sum_{k_1,k_2,k_3,k_4,k_5} \mathbb{E}a_{k_1}^2 a_{k_2} a_{k_3} a_{k_4} a_{k_5}\,, \\
\tilde{A} &= \sum_{k_1,k_2,k_3,k_4,k_5,k_6} \mathbb{E}a_{k_1} a_{k_2} a_{k_3} a_{k_4} a_{k_5} a_{k_6}\,.
\end{aligned}
$$

To avoid lengthy formula, we denote $\Delta_k W = W(x_{k+1}) - W(x_k)$, $\Delta_k = x_{k+1} - x_k$. We also omit the indices under the sigma notation. By the Cauchy-Schwarz inequality and (2.13)

$$
a_k^2 \le c_1^2(b-a)\frac{|\Delta_k W|^2}{\Delta_k}\int_{x_k}^{x_{k+1}} e^{2\theta|W_\pi(z)|}[L_B([\xi,\eta],S_{W_\pi}(z))-L_B([\xi,\eta],S_{W_\pi}(x_k))]^2 dz\,.
$$

Hence, A is bounded from the above by

$$
\begin{aligned}
c_1^6(b-a)\sum \mathbb{E}\int_{x_{k_1}}^{x_{k_1+1}} e^{2\theta|W_\pi(z)|}[L_B([\xi,\eta],S_{W_\pi}(z)) - L_B([\xi,\eta],S_{W_\pi}(x_{k_1}))]^2 \\
\frac{|\Delta_{k_1}W|^2}{\Delta_{k_1}} dz_1 a_{k_2} a_{k_3} a_{k_4} a_{k_5}\,.
\end{aligned}
$$

Taking the expectation with respect to the Brownian motion B first and applying (3.39) with $\beta = 1/2$ we see that A is bounded from the above by

$$
\begin{aligned}
&c_1^6(b-a)|\eta-\xi|^{3/2}\sum \mathbb{E}\int_{[x_k,x_{k+1}]} e^{2\theta|W_\pi(z_1)|+\theta|W(z_2)|+\cdots+\theta|W(z_5)|} \\
&\quad |S_{W_\pi}(z_1) - S_{W_\pi}(x_{k_1})|\prod_{j=2}^{5}|S_{W_\pi}(z_j) - S_{W_\pi}(x_{k_j})|^{1/2}\frac{|\Delta_{k_1}W|^2}{\Delta_{k_1}}\prod_{j=2}^{5}\frac{|\Delta_{k_j}W|}{\Delta_{k_j}} d\bar{z}\,,
\end{aligned}
$$

where $\int_{[x_k,x_{k+1}]} d\bar{z}$ denotes $\prod_{j=1}^5 \int_{x_{k_j}}^{x_{k_j+1}} dz_j$. We further apply the Hölder inequality and the simple estimate $\mathbb{E}e^{\theta|W_\pi(z)|} \le e^{\theta^2|z|/2}$. The above quality is bounded by a

constant multiple of

$$c_1^6(b-a)|\eta-\xi|^{3/2}\sum\int_{[x_k,x_{k+1}]}\left\{\mathbb{E}|S_{W_\pi}(z_1)-S_{W_\pi}(x_{k_1})|^2\prod_{j=2}^5|S_{W_\pi}(z_j)-S_{W_\pi}(x_{k_j})|\right\}^{1/2}$$
$$\times e^{\kappa(|z_1|+\cdots+|z_5|)}|\Delta_{k_j}|^{-1/2}d\bar{z}\,.$$

Applying the Hölder inequality again, we obtain

$$A\lesssim c_1^6(b-a)|\eta-\xi|^{3/2}\Delta\left(\int_a^b e^{\kappa|x|}dx\right)^5\,.$$

To estimate $\tilde{A}$, we first take the expectation with respect to the Brownian motion B. Using (3.41) with $\alpha\in[0,1]$ we have

$$\tilde{A}\lesssim c_1^6(b-a)|\eta-\xi|^{3\alpha}\sum\prod_{j=1}^6\int_{x_{k_j}}^{x_{k_j+1}}\mathbb{E}e^{\theta W_\pi(z_j)}|S_{W_\pi}(z_j)-S_{W_\pi}(x_{k_j})|^{1-\alpha}\,\frac{\Delta_{k_j}W}{\Delta_{k_j}}\,dz_j\,.$$

Applying the Hölder inequality yields

$$\tilde{A}\lesssim c_1^6(b-a)|\eta-\xi|^{3\alpha}\Delta^{6(\frac12-\alpha)}\left(\int_a^b e^{\kappa|x|}dx\right)^6\,. \tag{9.8}$$

Choosing α between 1/3 and 1/2 yields (9.3).

Proof of (9.4): From the Hölder inequality we have

$$\mathbb{E}|I_2([\xi,\eta])|^6\le(b-a)^5\times$$
$$\mathbb{E}\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|g(x,W_\pi(x))-g(x_k,W_\pi(x))|^6|L_B([\xi,\eta],S_{W_\pi}(x_k))|^6|\dot{W}_\pi(x)|^6dx\,.$$

An further application of the Hölder inequality, condition (2.14) and the estimate (3.39) with $\beta=0$ yields

$$\mathbb{E}|I_1([\xi,\eta])|^6\lesssim(b-a)^5c_2^6(b-a)e^{\kappa(|a|\vee|b|)}|\eta-\xi|^3\sum_{k=0}^{n-1}|x_{k+1}-x_k|^{6\lambda-2}\,,$$

which implies (9.4).

Proof of (9.5): Applying the moment inequality for martingales, we see that the expression on its left hand side is at most a constant times

$$\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\mathbb{E}\left[g(x,W(x))L_B([\xi,\eta],S_W(x))-g(x_k,W(x))L_B([\xi,\eta],S_{W_\pi}(x_k))\right]^6dx,$$

which is again bounded by the sum of a certain constant multiple of

$$D:=c_1^6(b-a)\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\mathbb{E}\left[L_B([\xi,\eta],S_W(x))-L_B([\xi,\eta],S_{W_\pi}(x_k))\right]^6e^{6\theta|W(x)|}dx,$$

and

$$\tilde{D}:=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\mathbb{E}\left[L_B([\xi,\eta],S_{W_\pi}(x_k))\right]^6\left[g(x,W(x))-g(x_k,W(x))\right]^6dx,$$

Similar to the estimation for I_2, it is easy to see that $\tilde{D}$ satisfies

$$\tilde{D} \lesssim c_2^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^3 \sum_{k=0}^{n-1} |x_{k+1} - x_k|^{6\lambda - 2}$$

which in turn satisfies the bound (9.5).

By mean of inequality (3.39) with $\beta \in (0, 1/2]$, D is bounded by a constant times

$$c_1^6 (b-a) |\eta - \xi|^{3(1-\beta)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} |S_W(x) - S_{W_\pi}(x_k)|^{6\beta} e^{6\theta |W(x)|} dx.$$

By the Hölder inequality, we see that above expression is at most a constant times

$$c_1^6 (b-a) |\eta - \xi|^{3(1-\beta)} |\Delta|^{3\beta} \int_a^b e^{\kappa |x|} dx \,,$$

which also yields (9.5).

Proof of (9.6): By the Hölder inequality, the quality on the left hand side of (9.6) is at most a constant times

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} \left[\partial_u g(x, W(x)) L_B([\xi, \eta], S_W(x)) - \partial_u g(x_k, W(x)) L_B([\xi, \eta], S_{W_\pi}(x_k)) \right]^6 dx \,.$$

From here, (9.6) follows similarly. $\square$

9.2. **Convergence over $\mathbb{R}$.** Let γ and κ be the constants in Lemma 9.1. Let π be a partition of $\mathbb{R}$. For every $N \in \mathbb{Z}$, let π_N be the partition on $[N-1, N]$ induced by π and $|\pi_N|$ denote the mesh size of π_N. For every $\delta > 0$, we now choose a partition $\pi(\delta)$ such that

$$\sum_N c_3([N-1, N]) (e^{\kappa |N|} |\pi_N|^\gamma)^{\frac{1}{6}} \le \delta \,. \tag{9.9}$$

With the notations in the previous subsection, the process $\ell_\pi(g, \cdot)$ (defined in (9.1)) can be written as

$$\ell_\pi(g, \xi) = \sum_{N \in \mathbb{Z}} \ell_{\pi_N}^{[N-1, N]}(g, \xi) \,,\ \xi \ge 0, \tag{9.10}$$

where $\ell_{\pi_N}^{[N-1,N]}(g, \cdot)$ is the process defined in (9.2). Finiteness of the process $\ell_\pi(g, \cdot)$ will become clear at the end of this subsection. For a random variable Y, we denote the L^6-norm $\|Y\|_6 := (\mathbb{E} Y^6)^{1/6}$. To simply notations, we further denote

$$\ell(g, [\xi, \eta]) = \int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x)$$

and

$$\ell^{[N-1,N]}(g, [\xi, \eta]) = \int_{N-1}^{N} g(x, W(x)) L_B(\xi, S_W(x)) W(d^o x) \,.$$

From the estimate (9.7), we obtain

$$\begin{aligned} \|\ell_\pi(g, [\xi, \eta]) - \ell(g, [\xi, \eta])\|_6 &\le \sum_{N \in \mathbb{Z}} \left\| \ell_{g, \pi_N}^{[N-1,N]}(g, [\xi, \eta]) - \ell^{[N-1,N]}(g, [\xi, \eta]) \right\|_6 \\ &\lesssim |\eta - \xi|^{(1+\epsilon)/6} \sum_{N \in \mathbb{Z}} c_3([N-1, N]) \left(|\pi_N|^\gamma e^{\kappa |N|} \right)^{1/6} \,. \end{aligned}$$

We now choose $\pi = \pi(\delta)$ and use the condition (9.9) to obtain

$$\|\ell_{\pi(\delta)}(g,[\xi,\eta]) - \ell(g,[\xi,\eta])\|_6 \lesssim |\eta-\xi|^{(1+\epsilon)/6}\delta\,. \tag{9.11}$$

Let K be any positive number. Applying the Garsia-Rodemich-Rumsey inequality (see [7]), wee see that there exists a continuous version of the process $\ell_\pi(g,\cdot)-\ell(g,\cdot)$ which satisfies the following estimate almost surely

$$\sup_{0<\xi<\eta<K} \frac{|\ell_{\pi(\delta)}(g,[\xi,\eta]) - \ell(g,[\xi,\eta])|}{|\eta-\xi|^{\epsilon/8}} \le C_K\delta\,. \tag{9.12}$$

Since $\ell(g,\cdot)$ has a continuous version and is finite almost surely, this implies the same properties holds for $\ell_{\pi(\delta)}(g,\cdot)$. Moreover, we have also proved the uniform convergence

$$\lim_{\delta\to0}\sup_{0<\xi<\eta<K} \frac{|\ell_{\pi(\delta)}(g,[\xi,\eta]) - \ell(g,[\xi,\eta])|}{|\eta-\xi|^{\epsilon/8}} = 0. \tag{9.13}$$

which holds almost surely. This finishes the proof of Step 2, and hence of Proposition 3.5.

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